Tomographic Inversion using NURBS and MCMC

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***Abstract*—A new approach in tomographic inversion using Non-Uniform Rational B-Splines (NURBS) combined with Markov Chain Monte Carlo is discussed. Low dimension of parameters is the benefit in using NURBS, but the resulting inverse problem is nonlinear. Markov Chain Monte Carlo comes forth to tackle this problem. Another advantage is that the result will be directly in CAD software so that it will be convenient for optimizing the shape. Numerical example with simple simulated data, a simple homogeneous simple shape with attenuation one inside the curve and zero outside the curve is given. The result is compared with filtered back projection and Tikhonov regularization. The potential drawback of the proposed method is heavy computation.**

Keywords - tomographic; NURBS; Bayesian inversion; MCMC

I. Introduction

Tomography is a useful way to study unknown structures of the object. In tomography, the measurement data of the object are collected from various directions. One example is X-ray tomography, based on the absorption of X-rays as they pass through the different parts of an object. Another example is an electron microscopy (EM), which uses a beam of electrons to create an image of a specimen and produce a magnified image.

In some applications of tomography, the dataset is limited. One example is in the medical environment, where it is important to avoid a high X-ray dose to the patient. This situation leads to the production of only sparse data. In EM, the specimen cannot be tilted in all directions, leading to a limited-angle problem, and making the reconstruction task is very ill-posed (i.e. extremely sensitive to measurement noise and modeling error). Therefore, a process of introducing additional information in order to solve this problem is needed. Therefore, some additional information needs to be introduced for making the recovery process reliable and robust against noise. Such information is called *a priori* knowledge. Commonly, a penalty for complexity term emerges on the basis of the information.

Here, we propose a new approach for tomographic reconstruction from sparse data. The unknown parameter is modeled using Non-uniform rational B-splines (NURBS). In 2D, the curve is determined by a parameter vector called the knot vector and some points called control points. There are advantages in using NURBS;

- The number of parameters is small, because we only need to recover the control points, which are very few in number compared to the points of the curve. Having fewer parameters lead to more robust algorithms.
- By working in NURBS, the results are readily in a form used by computer-aided design (CAD) software, because NURBS is the building block of CAD systems and the computer numerical control (CNC) machines of industrial production devices use NURBS to describe the shapes of objects to be manufactured. Therefore, it is will be convenient in creating, modifying, analyzing, or optimizing. Most modern factories use CNC machines, and relatively cheap (less than USD 5000) devices are available for small-scale production. Having the result of reconstruction immediately in an industrially producible form can save time in research and development and enable new kinds of production possibilities for start-up companies located anywhere in the world, including developing countries.

The difficulty in the proposed method is that the linear inverse problem of tomography becomes nonlinear (observing only the control points). Nonlinear inverse problem have more complex relationships between the data and model. This is why we use the flexible computational approach called Bayesian inversion [2].

Statistical (Bayesian) inversion provides an effective way to complement insufficient and incomplete measurements with *a priori* knowledge of the unknown. In any particular application there is typically some understanding about the types of objects one is looking for. Markov Chain Monte Carlo is one of the most popular techniques for sampling from probability distributions, and is based on the construction of a Markov chain that has the desired distribution as its equilibrium distribution. With the increasing availability of computer power, Monte Carlo techniques are being increasingly used. Monte Carlo methods are especially useful for simulating systems with many coupled degrees of freedom.

This paper presents the first feasibility study on the new approach. In this preliminary analysis, we use simulated data and a relatively simple problem, but the reconstruction works successfully. This approach is quite promising for solving more complex problems.

II. NURBS

NURBS is a parametric representation for describing and modeling curves and surfaces which are basically piecewise polynomial functions. NURBS curve and surface is a standard system in CAD because NURBS model is powerful and flexible.

There are 3 important things in NURBS, those are control points, knots, and basis functions.

1) Control Points ($\mathbf{p}_i$)
Control points are a set of points by which the positions can determine the shape of NURBS curves. The curve can be managed easier by connecting the control points by the line sequentially, called *control polygon*. The shape of control polygon will be followed by the curve.

2) Knots Vector
A knot vector is a set of coordinates in the parametric space. In one dimension knot vector is written

$$\mathbf{t} = \{t_0, t_1, ..., t_{n+p+1}\},$$

where $t_i \in$ R is the ith knot, $i = 1, 2, ..., n + p+1$ is the knot index, n is the number of basis function and p is the order of polynomial function. The curve is divided into interval by this vector and the knot vector gives information how width the interval affected by the changing of control points. Insertion and removing knots are possible to handle the curve in the proper space.
Basically, there are two types of knot vector:
- *uniform* if the knots are equally-spaced in the parametric space.
- *nonuniform* knot vectors may have either spaced and/or multiple internal knot elements.

If the first and last knot vectors element appear $p+1$ times, then it is called *open* knot vector, otherwise it is called *periodic* knot vector. The knot vector has to be monotonically increasing as below:

$$t_{i-1} \leq t_i, \quad \forall i = 1, 2, ..., (n + p + 1).$$

Formally, an open *uniform* knot vector is given by

$$t_i = 0, \quad 1 \leq i \leq p$$
$$t_i = i - p, \quad p + 1 \leq i \leq n + 1$$
$$t_i = n - p + 2, \quad n + 2 \leq i \leq n + p + 1.$$

3) Basis Function
Basis Function, $N_{i,p}(t)$, is a function that gives information how strongly the ith control point, $\mathbf{p}_i$, attracts the curve in specific interval, where i is index of control point.
The basis functions have the following form:

$$N_{i,0}(t) = \begin{cases} 1 & \text{if } t_i \leq t < t_{i+1}, \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{aligned} N_{i,p}(t) &= \frac{t - t_i}{t_{i+p} - t_i} N_{i,p-1}(t) + \\ & \frac{t_{i+p+1} - t}{t_{i+p+1} - t_{i+1,p-1}} N_{1+i,p-1}(t). \end{aligned} \quad (1)$$

The general form of NURBS curve can be written as follow

$$\mathcal{S}(t) = \sum_{i=0}^{n} p_i^h N_{i,p}(t),$$

where the $p_i^h s$ are the four-dimensional homogenous control polygon vertices for the non-rational form-dimensional B-Spline curve.

From (1), the four-dimensional space is projected back into three-dimensional space by dividing with the homogeneous coordinate which yields the rational B-Spline curve as follow:

$$\begin{aligned} \mathcal{S}(t) &= \frac{\sum_{i=0}^{n} \mathbf{p}_i N_{i,p}(t)\omega_i}{\sum_{i=0}^{n} N_{i,p}(t)\omega_i} \\ &= \sum_{i=0}^{n} \mathbf{p}_i R_{i,p}(t), \end{aligned} \quad (2)$$

where $\mathbf{p}_i$s are the three dimensional control points for the rational B-spline curve, ω_i are the weights, and the

$$R_{i,p}(t) = \frac{\omega_i N_{i,p}(t)}{\sum_{i=0}^{n} \omega_i N_{i,p}(t)}, \quad (3)$$

are the rational B-splines basis function. The $\omega_i \geq 0$ for all i. The weight can be one of the controller in attracting the curve to the control points. A curve with all weights which set to 1 has the same shape as if all weights set to 10. The shape of different curve will change if the weights of control points are different, while other elements are fixed as in [5], [6].

Basically, the shape of NURBS curve is defined by the knot vectors and location of the control points and the weights. Most designers assume that the knot vectors are fix and only allow to modify the control points and the weights. In this simulation data, the weights are assumed to be equal. The open uniform knot vector is chosen as a common knot vector used in CAD.

NURBS has several important qualities which make it powerful for modeling. NURBS provides the flexibility to design many variety of shapes (standard analytic shapes and free-form shape of curves and surfaces) by manipulating the control points and the weights. The amount of information or parameters required for a NURBS representation is much smaller than the amount of information required by other common representations. Those conditions bring to the evaluation which is reasonably fast and computationally stable. Invariant under scaling, rotation, translation, and shear as well as parallel and perspective projection are also interesting and important properties of NURBS curve.

III. Tomographic Measurement Model

In this preliminary result we measure the simple shape. We build homogeneous simple bottle shape from closed NURBS curve.

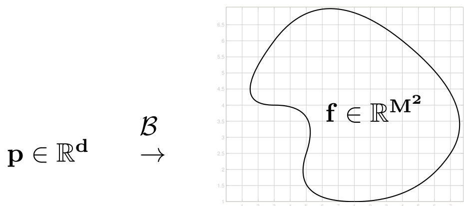

Figure 1. Mapping from $\mathbf{p}$ to $\mathbf{f}$

To avoid inverse crime [4], we produce the synthetic phantom using NURBS with 10 parameters and knot vector

$$[0\ 0\ 0\ \frac{1}{10}\ \frac{2}{10}\ \frac{3}{10}\ \frac{4}{10}\ \frac{5}{10}\ \frac{6}{10}\ \frac{7}{10}\ \frac{8}{10}\ \frac{9}{10}\ 1\ 1\ 1],$$

and equal weight for all control points. In the inversion, eight parameters are recovered with different open knot vector.

From the NURBS curve, we set the X-ray attenuation becomes one inside the curve and zero outside the curve.

Consider the operator $\mathcal{B}$ which has the following form:

$$\mathcal{B}(\mathbf{p}) = \begin{cases} 1 \text{ if pixel inside the NURBS curve,} \\ 0 \text{ if pixel outside the NURBS curve.} \end{cases}$$

The vector $\mathbf{f}$ comes from the mapping of the parameter $\mathbf{p}$ applied to operator $\mathcal{B}$.Our proposed shape is as Fig. 2.

The object is measured with pixel size 64×64 using parallel beam geometry, as an example see Fig. 3.

From the source, the wave will be penetrated through the matter and the detector will record the projection images from different directions. All X-ray imaging are based on the absorption of X-ray as they pass through the different parts of the object.

In this simulated data, we collected projection of the images of the object from 18 directions (i.e. the measured angle are $0°, 10°, 20°, ..., 170°$) and using 95 lines for each direction.

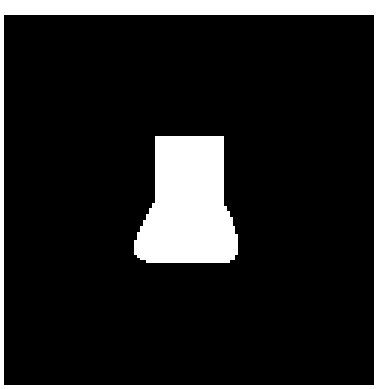

Figure 2. Homogeneous simple shape NURBS

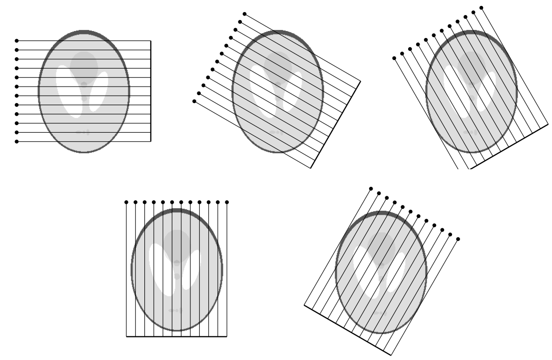

Figure 3. Parallel beam X-ray measurement geometry. There are 5 different directions (angles) and 11 lines. Black dots show the locations of the X-ray source at different times of measurement. The thick line represents the detector measuring the intensity of the X-rays after passing through the target. High attenuation is shown here as darker shade of gray and low attenuation as lighter shade.

IV. Bayesian Inversion

Here we construct the measurement model as follow, consider the indirect measurement

$$m = A(\mathbf{f}) + \varepsilon,$$

where $m \in \mathbb{R}^k$ is measurement data, A is an operator of projection image from Radon transform where $\mathbf{f}$ is quantity of interest and ε is the errors of the measurement.

The inverse problem is to find $\mathbf{f}$ which depends on $\mathbf{p}$, the parameters (control points) of the NURBS curve.

We use probability theory to model our lack of information in the inverse problem. Markov Chain Monte Carlo method can be used to generate the parameters according to the conditional probability

$$\pi(\mathbf{p} \mid \mathbf{m}) = \frac{\pi(\mathbf{p})\pi(\mathbf{m} \mid \mathbf{p})}{\pi(\mathbf{m})},$$

called the *posterior distribution* and with the *likelihood function*

$$\pi(m \mid \mathbf{p}) = \mathbf{C} \exp(-\frac{\mathbf{1}}{\mathbf{2}\sigma^{\mathbf{2}}} \|\mathbf{A}(\mathcal{B}\mathbf{p}) - \mathbf{m}\|_{\mathbf{2}}^{\mathbf{2}}).$$

Assume that the angle of each parameter is not less than $(i-1)45$ and not more than $(i+1)45$, where i is the index of parameter and the radius of each parameter from the central point of the object is not less than 0 and not more than 15. This information becomes our *prior* information and it guarantees that the behavior of the parameters will not switch each other during the computation. We formulate the angles condition as follow

$$\mathbf{A}(\theta_{\mathbf{i}}) = \begin{cases} 1 - \frac{|\theta_i - \theta_i'|}{45} & \text{for } \theta_i' - 45 \le \theta_i \le \theta_i' + 45 \\ \\ 0, & \text{otherwise,} \end{cases}$$

where $\theta_i' = 45i$.

The radius terms is as follow

$$\mathbf{R}(r_i) = \begin{cases} 1 - \frac{|r_i - 1|}{15} & \text{for } 0 \leq r_i \leq 15 \\ 0, & \text{otherwise.} \end{cases}$$

Hence,

$$\pi(\mathbf{p}) = \mathbf{A}(\theta_i) \cdot \mathbf{R}(r_i)$$

as our *prior* distribution.

To get a useful answer to our inverse problem, we need to draw a representative estimate from the posterior probability distribution. We study the *conditional mean estimate* (CM) defined as the integral

$$\mathbf{p}^{CM} := \int_{\mathbb{R}^N} \mathbf{p}\pi(\mathbf{p} \,|\, \mathbf{m})\, \mathbf{dp}. \tag{4}$$

However, the integration in (4) is over a high-dimensional space, and standard numerical integration quadratures are ineffective. We resort instead to Markov chain Monte Carlo (MCMC) methods, whose basic idea is to generate a random sequence $p^{(1)}, p^{(2)}, \ldots, p^{(N)}$ of samples with the property that

$$\mathbf{p}^{CM} \approx \frac{1}{N} \sum_{k=1}^{N} p^{(k)}, \tag{5}$$

and denote

$$\mathbf{p}_N^{CM} = \frac{1}{N} \sum_{k=1}^{N} p^{(k)}. \tag{6}$$

The sequence $p^{(1)}, p^{(2)}, \ldots, p^{(N)}$ of vectors can, of course, be analyzed more thoroughly than just by taking their average. In recent years, statisticians have been increasingly drawn to MCMC methods to simulate nonstandard multivariate distributions. The Gibbs sampling algorithm is one of the best known of these methods but a considerable about of attention is now being devoted to the Metropolis-Hasting algorithm. We use the Metropolis-Hastings algorithm to get the parameters sequences. The algorithm takes the form:

1) Set $n = 1$ and initialize $\mathbf{p}^{(1)}$, where $\mathbf{p}^{(1)}$ depends on $\theta^{(\mathbf{1})}$ and $\mathbf{r}^{(\mathbf{1})}$.
2) Draw a random integer k from 1 to number of control points.
3) Set $\theta := \theta^k + \epsilon_k$ and $r := r^k + \epsilon_k$. Set $p^{(k)} = (r\cos\theta, r\sin\theta)$ then $\mathbf{p}$ will contain the proposed $p^{(k)}$.
4) If $\pi(\mathbf{p}|m) \geq \pi(\mathbf{p}^{(n)}|m)$ then set $\mathbf{p}^{(n+1)} := \mathbf{p}$.
5) If $\pi(\mathbf{p}|m) < \pi(\mathbf{p}^{(n)}|m)$, draw a random number s from uniform distribution on $[0, 1]$.
   If $s \leq \frac{\pi(\mathbf{p}|m)}{\pi(\mathbf{p}^{(n)}|m)}$, then set $\mathbf{p}^{(n+1)} := \mathbf{p}$; else set $\mathbf{p}^{(n+1)} := \mathbf{p}^{(n)}$.
6) If $n = N$ then stop; else set $n := n + 1$ and go to 2^{nd} step.

Applying the CM estimate parameter, $\mathbf{p}_N^{CM}$ to the NURBS curve (2), we denote by $\mathcal{S}_N^{CM}$. Then apply it to operator $\mathcal{B}$, we get the reconstruction of the shape,

$$\mathbf{f}_N^{CM} = \mathcal{B}(\mathbf{p}_N^{CM}).$$

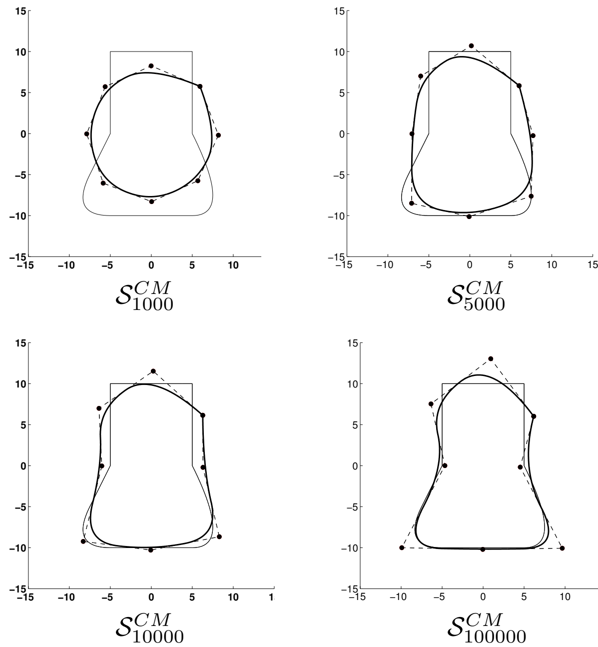

Figure 4. The thin black line is the target curve. The thick black line is the reconstruction of the NURBS curve, $\mathcal{S}_N^{CM}$. The black circle markers are control points, $\mathbf{p}_N^{CM}$.

V. Computational Result

We present numerical example to demonstrate our proposed method with 8 parameters of interest. We fix the knot data in each iteration becomes the open uniform knot vector order 3:

$$[0\ 0\ 0\ \frac{1}{7}\ \frac{2}{7}\ \frac{3}{7}\ \frac{4}{7}\ \frac{5}{7}\ \frac{6}{7}\ 1\ 1\ 1].$$

The object is measured by Radon transform using 0.1% noise. Using Metropolis-Hasting algorithm we get the average of the parameters's chains for the reconstruction of the curve for some iterations as Fig. 4.

The iteration stop when $N = 1000000$. The average of the control points, $\mathbf{p}_N^{CM}$, is as Table I. By applying the control points to the NURBS curve (2), then we get the curve $\mathcal{S}_N^{CM}$ as Fig. 5. The effect choosing the open knot vector and the first control point as the last control point yields the kink in this point as in Fig. 4. Finally, using the operator $\mathcal{B}$, we get the shape reconstruction as Fig. 6.

The reconstruction using filtered back projection and Tikhonov regularization also given. In filtered back projection, the object is recovered by using *iradon* command in MATLAB as it is shown in Fig. 7. The Tikhonov regularization is discussed in [4]. See Fig. 8. Both of the reconstruction use pixel size 64×64.

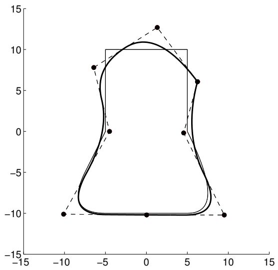

Figure 5. The thin black line is the target curve. The thick black line is the reconstruction of the NURBS curve, $\mathcal{S}_{1000000}^{CM}$. The black circle markers are control points, $\mathbf{p}_{1000000}^{\mathbf{CM}}$.

Computation time for all method are recorded as in Table II.

VI. Discussion and Conclusion

In Fig. 6 we see the reconstruction using our proposed method in image size 512×512. Changing the resolution in this case does not matter because our reconstruction is in vector graphics form. Only with 16 numbers (Table I), this NURBS-MCMC reconstruction can recover the data successfully and it is automatically in CAD format or CNC machine.

In filtered backprojection and Tikhonov reconstruction, we need 4096 numbers to give us the information of the object. Also we can see from Fig. 7 and Fig. 8 that it is almost impossible to represent the shape of the object because there are so many artifacts appearing. Because of this, these reconstructions cannot represent the result directly in CNC machine. It is very different with NURBS-MCMC reconstruction, which only has exactly two colors, black and white, hence the shape of the object is obvious.

The proposed method, NURBS-MCMC, is quite promising to be applied in computational tomography inversion. The potential drawback of this method is heavy in computation as we can see Table II, but this drawback can be solved by using parallel computing.

Table I
EIGHT CONTROL POINTS, $\mathbf{p}_{1000000}^{\mathbf{CM}}$, ARE OBTAINED FROM THE NURBS-MCMC RECONSTRUCTION

x	y
6.24	6.07
1.33	12.7
-6.24	7.80
-4.50	0
-10.1	-10.1
0.03	-10.2
9.50	-10.2
4.55	-0.18

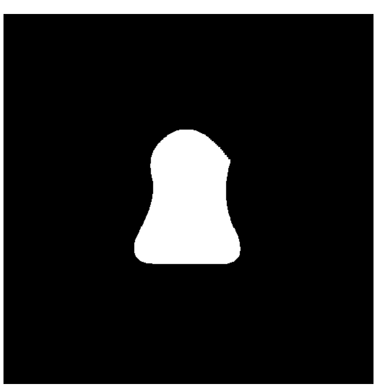

Figure 6. Final reconstruction using NURBS and MCMC in image size 512×512.

Table II
CONSUMING TIME FOR ALL RECONSTRUCTION METHODS

FBP	Tikhonov regularization	NURBS-MCMC
1.5 s	905 s	17 000 s

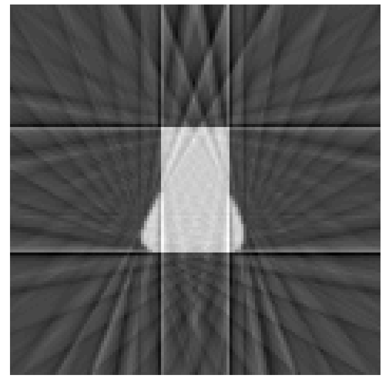

Figure 7. Reconstruction using filtered back projection with error 0.1%.

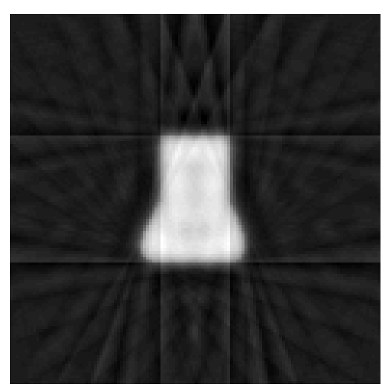

Figure 8. Reconstruction using Tikhonov Regularization with error 0.1%.

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