Reconstruction of High Contrast 2-D Conductivities by the Algorithm of A. Nachman

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ABSTRACT. The uniqueness proof of A. Nachman [20] for the 2D inverse conductivity problem outlines a reconstruction algorithm for determining the unknown conductivity in a region $\Omega$ from knowledge of the Dirichlet-to-Neumann map. The algorithm is a direct method based on the techniques of inverse scattering. Here we present a practical implementation of the algorithm in Nachman’s proof and demonstrate its effectiveness on several radially symmetric conductivities in $C^3(\Omega)$ and $C^4(\Omega)$.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected $C^\infty$ domain and $\gamma \in C^2(\Omega)$. We assume that $\gamma(x) \geq c > 0$ and $\gamma \equiv 1$ in a neighborhood of $\partial \Omega$.

Define the Dirichlet-to-Neumann map corresponding to $\gamma$ by

\begin{equation}
\Lambda_\gamma : H^{1/2}(\partial \Omega) \to H^{-1/2}(\partial \Omega), \quad \langle \Lambda_\gamma f, g \rangle = \int_{\partial \Omega} \gamma \nabla u \cdot \nabla v,
\end{equation}

where $v$ is any $H^1(\Omega)$ function with trace $g$ on the boundary and $u$ is the unique $H^1(\Omega)$ solution of the Dirichlet problem

\begin{equation}
\begin{cases}
\nabla \cdot \gamma \nabla u &= 0 \text{ in } \Omega, \\
\quad u &= f \text{ on } \partial \Omega.
\end{cases}
\end{equation}

The inverse conductivity problem of Calderón [7] is to decide whether $\gamma$ is uniquely determined by $\Lambda_\gamma$ and if so, reconstruct $\gamma$ from the knowledge of $\Lambda_\gamma$. Physically, $u$ represents the electric potential and $\gamma$ the conductivity. Knowledge of the Dirichlet-to-Neumann map is tantamount to knowing the resulting current pattern on $\partial \Omega$ corresponding to any prescribed voltage pattern on $\partial \Omega$. The inverse problem has applications in geophysics, nondestructive testing and a medical imaging technique known as Electrical Impedance Tomography (EIT). In EIT the domain $\Omega$ is often a cross-section of the body, such as a patient’s chest. A basis of current patterns is applied on electrodes attached around the patient’s chest and the resulting voltages are measured on the electrodes. Since the tissues and organs in the body have

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different conductivities, plotting the conductivity distribution $\gamma(x)$ yields a 2-D image of a cross-section of the chest. See [9] for a recent survey article on EIT.

The inverse conductivity problem is ill-posed in the sense that large changes in the conductivity can correspond to small changes in the boundary data. Furthermore, the problem is more difficult in dimension $n = 2$ than $n \geq 3$. One can see this formally by considering the Schwartz kernel of the data. The kernel contains $2(n - 1)$ degrees of freedom, so in 2-D the problem is formally determined, while in $n \geq 3$ it is overdetermined. Thus in 2-D one must in some sense make use of all of the data.

We briefly discuss some of the milestone results for the inverse conductivity problem, with no claim to completeness. Motivated by geophysical applications, in 1980 Calderón [7] showed how to determine nearly constant conductivities from the Dirichlet-to-Neumann map. In 1985 Kohn and Vogelius [17] proved that if $\partial \Omega$ is $C^\infty$ and $\gamma$ is piecewise analytic, then $\Lambda_\gamma$ determines $\gamma$ uniquely in dimensions $n \geq 2$. Global uniqueness for $n \geq 3$ with $\gamma$ in $C^\infty(\Omega)$ and $C^\infty \partial \Omega$ was established by Sylvester and Uhlmann [24]. Nachman [19] gave a reconstruction method for dimensions $n \geq 3$ for $\gamma \in C^{1,1}$ with $\partial \Omega \in C^{1,1}$. The global uniqueness question in 2-D remained open until 1995, when it was resolved for $\partial \Omega$ Lipschitz and $\gamma \in W^{2,p}(\Omega), p > 1$, by Nachman [20]. This result was sharpened in 1997 to $W^{1,p}(\Omega), p > 2$, conductivities by Brown and Uhlmann [5]. For generalizations of the above results to more general spaces and other related work see the references given in [25, 20, 15].

An important feature of Nachman’s proof is that it is constructive; it outlines a direct method for solving for $\gamma$ without iteration. Previous algorithms have relied on iterative techniques such as least squares minimization (see, for example, [10], [16], [6]), linearization ([7, 2, 8]), or layer-stripping ([22], [23]). Nachman’s proof is based on the techniques of inverse scattering and the $\partial$ method. The $\partial$ method was first used by Beals and Coifman [3] for the quantum inverse scattering problem in 1-D and was extended to 2-D problems in [4]. It makes use of the Green’s function for the Laplacian introduced by Faddeev [11] and the corresponding exponentially growing solutions to the Schrödinger equation. In the $\partial$ approach a nonphysical function not directly measurable in experiments known as the scattering transform $t(k)$ plays a key role. In [20] an integral equation is derived for determining the scattering transform from the Dirichlet-to-Neumann data. The scattering transform is a function in the $\partial$ equation, which must be solved to obtain the exponentially growing solutions in the complex plane. In the final step, the conductivity $\gamma$ is recovered by taking the small $k$ limit of the solution to the $\partial$ equation.

In this work we review the reconstruction algorithm set forth in Nachman’s proof and the implementation developed in [21]. In [21] the algorithm was tested on two radially symmetric $C^\infty$ conductivities, one low-contrast example with height 1.2 and one high-contrast example with height 4. Here we consider three radially symmetric $C^3$ examples, with heights approximately 4, 6, and 8, and we study a slightly more complicated $C^4$ example. These new examples provide insight into the relationship between the conductivity and the scattering transform and demonstrate the effectiveness of the algorithm on less regular and higher contrast conductivities. The $C^4$ example dips below the boundary value $\gamma = 1$ in region near $\partial \Omega$. Such an example can be viewed as the first step in simulating physically relevant conductivities, which would have features such as a resistive region (corresponding to skin) near the boundary.
2. The reconstruction algorithm

In this section we give a brief outline of the reconstruction algorithm set forth in Nachman’s uniqueness proof [20].

First, the conductivity equation (1.2) is transformed to the Schrödinger equation by the following change of variables. Let $q \in C^2_0(\Omega)$ be given by

$$ q = \gamma^{-1/2} \Delta \gamma^{1/2}. $$

If $u$ is a solution of $\nabla \cdot \gamma \nabla u = 0$ in $\Omega$, defining $\tilde{u} = \gamma^{1/2} u$ yields

$$ (-\Delta + q) \tilde{u} = 0 \quad \text{in} \quad \Omega. \quad (2.1) $$

While in [20] the first step is to find $\gamma|_{\partial \Omega}$ and $\partial / \partial n|_{\partial \Omega}$ and to continue $\gamma$ artificially to be one outside a neighborhood of $\Omega$, we omit that step in this work and consider only conductivities that are one in a neighborhood of $\partial \Omega$. This assumption allows us to smoothly extend $\gamma = 1$ and $q = 0$ to the whole plane. We may therefore study equation (2.1) in all of $\mathbb{R}^2$.

The exponentially behaving solutions of (2.1) introduced by Faddeev [11] are the key to the reconstruction. By Theorem 1.1 of [20] for any $k \in \mathbb{C} \setminus 0$ there is a unique solution $\psi(x, k)$ of

$$ (-\Delta + q) \psi(x, k) = 0 \quad x \in \mathbb{R}^2 \quad (2.2) $$

satisfying $e^{-ikx} \psi(\cdot, k) - 1 \in W^{1, \hat{p}}(\mathbb{R}^2)$ for any $2 < \hat{p} < \infty$. The space $W^{1, \hat{p}}(\mathbb{R}^2)$ is a special case of the definition

$$ W^{m, \rho}(E) = \{ f \in L^\rho(E) \mid \partial^\alpha f \in L^\rho(E), \ |\alpha| \leq m \} $$

for an arbitrary domain $E \subset \mathbb{R}^n$ and $1 \leq \rho \leq \infty, m \geq 0$.

Denote

$$ \mu(x, k) := e^{-ikx} \psi(x, k), \quad x \in \mathbb{R}^2, k \in \mathbb{C} \setminus 0. \quad (2.3) $$

Here $ikx = i(k_1 + ik_2)(x_1 + ix_2)$. Then $\mu$ satisfies

$$ (-\Delta - 2i k \tilde{\partial} + q) \mu = 0 \quad (2.4) $$

where $\tilde{\partial} = (\partial / \partial x_1 + i \partial / \partial x_2)/2$. The condition $\mu - 1 \in W^{1, \hat{p}}$ and the Sobolev imbedding theorem yield that $\mu$ is continuous and tends to one asymptotically when $|x| \to \infty$.

The reconstruction of $\gamma$ is based on the use of an intermediate object called the non-physical scattering transform $t$, which is not directly measurable in experiments:

$$ t(k) := \int_{\mathbb{R}^2} e_k(x) \mu(x, k) q(x) dx, \quad k \in \mathbb{C} \setminus 0, \quad (2.5) $$

where $e_k(x) := \text{exp}(i(kx + k \tilde{x}))$. Note that since $\mu$ is asymptotically close to one, $t(k)$ is approximately the Fourier transform of $q(x)$ evaluated at the point $(-2k_1, 2k_2) \in \mathbb{R}^2$.

The direct reconstruction method consists of two main steps:
1. Given $\Lambda_1$, determine the scattering transform $t(k)$.
2. Determine $\gamma$ from the knowledge of $t(k)$.

To obtain $t(k)$ from the Dirichlet-to-Neumann data, one must first solve an integral equation for the trace on $\partial \Omega$ of the exponentially growing solution $\psi$. Denote the Dirichlet-to-Neumann map of the homogeneous conductivity 1 by $\Lambda_1$ and note that since $\gamma \equiv 1$ near $\partial \Omega$ the maps $\Lambda_1$ and the Dirichlet-to-Neumann
map $\Lambda_\gamma$ of the Schrödinger problem are the same. By Theorem 5 of [20] for any $k \in \mathbb{C} \setminus 0$ the following integral equation is uniquely solvable on $H^{1/2}(\partial \Omega)$.

\[(2.6)\quad \psi(\cdot, k)|_{\partial \Omega} = e^{ikx} - S_k(\Lambda_\gamma - \Lambda_1)\psi(\cdot, k)\]

where

\[(2.7)\quad (S_k\phi)(x) := \int_{\partial \Omega} G_k(x - y)\phi(y)d\sigma(y),\]

and $G_k(x)$ is the Faddeev Green's function defined by

\[(2.8)\quad G_k(x) := e^{ikx}g_k(x), \quad -\Delta G_k = \delta,\]

\[(2.9)\quad g_k(x) := \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{e^{ix\xi}}{\xi(\xi + 2k)} d\xi, \quad (-\Delta - 4ik\bar{\partial})g_k = \delta.\]

Now $t(k)$ can be recovered from the formula

\[(2.10)\quad t(k) = \int_{\partial \Omega} e^{ik\bar{\xi}}(\Lambda_\gamma - \Lambda_1)\psi(\cdot, k)d\sigma.\]

To determine $\gamma$ from $t(k)$, one must first find $\mu(x, k)$ for all $k \in \mathbb{C} \setminus 0$. Namely, Theorem 2.1 of [20] implies that the $\bar{\partial}$ equation

\[(2.11)\quad \frac{\partial}{\partial k}\mu(x, k) = \frac{1}{4\pi k} t(k)e^{-k(x)}\mu(x, k), \quad k \neq 0,\]

holds and Theorem 4.1 of [20] shows that (2.11) is uniquely solvable. The solution satisfies the Fredholm integral equation

\[(2.12)\quad \mu(x, k) = 1 + \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{t(k')}{(k - k')^2}e^{-z(k')\mu(x, k')}dk'dk'\]

for all $k \in \mathbb{C} \setminus 0, x \in \mathbb{R}^2$. Note that the integral is taken over the $k$-plane, so to solve (2.12) $\mu(x, k)$ is needed for all values of $k \in \mathbb{C} \setminus 0$. Also note that in the solution of the $\bar{\partial}$ equation $x$ is kept fixed, so the computations can be carried out only in the region of interest.

One can recover the conductivity $\gamma(x)$ from the function $\mu(x, k)$. Formally, since

\[(-\Delta + q)e^{ikx}\mu(x, k) = 0,\]

taking $k = 0$ implies

\[(-\Delta + \frac{\Delta\gamma^{1/2}}{\gamma^{1/2}})\mu(x, 0) = 0\]

and we see that

\[\gamma^{1/2}(x) = \lim_{k \to 0} \mu(x, k).\]
3. From $\Lambda, \gamma$ to $t(k)$

In this section we describe how we obtain the scattering transform from the Dirichlet-to-Neumann map numerically. In [18] it is shown that the step from $\Lambda, \gamma$ to $t(k)$ has only a logarithmic stability estimate while the step from $t(k)$ to $\gamma$ has linear stability.

In [21] it is shown that if $\gamma(x)$ is rotationally symmetric, so is $t(k)$. In this work we consider radial conductivity distributions and make use of this fact in our construction of the Dirichlet-to-Neumann data and in our approximation of $t(k)$. In the numerical solution of the $\tilde{\delta}$ equation, no radially symmetry of $\gamma$ or $t(k)$ is assumed. If $\gamma(x) = \gamma(|x|)$ we know from [23] that the functions $\phi_n(\theta) := (2\pi)^{-1/2} e^{in\theta}$, $n \in \mathbb{Z}$, are eigenfunctions for $\Lambda, \gamma$. Thus $\Lambda, \gamma$ can be represented in the trigonometric basis by the collection $\{\lambda_n\}_{n=-\infty}^{\infty}$ of its eigenvalues.

To produce numerical data, we need to approximate these eigenvalues numerically. As explained in [12], we note that if two radial $L^\infty$ conductivities $\gamma$ and $\tilde{\gamma}$ satisfy $\gamma(x) \leq \tilde{\gamma}(x)$ pointwise in $\Omega$, the eigenvalues satisfy $\lambda_n \leq \tilde{\lambda}_n$. Moreover, for piecewise constant radial conductivities we can compute the eigenvalues explicitly [21]. Thus approximating the $C^2$ conductivity from above and below with piecewise constant conductivities gives us numerical upper and lower bounds for the eigenvalues. See [21] for more details.

To obtain $t(k)$ from $\{\lambda_n\}$ numerically, we use the approximation $\psi(x,k)|_{\partial \Omega} \approx e^{ikx}$, as opposed to solving the integral equation (2.6). Expanding $e^{ikx}$ in a Fourier series on the circle $x = e^{i\theta}$ yields [14]

$$e^{ikx} = \sum_{n=-\infty}^{\infty} a_n(k)e^{in\theta} \quad \text{with} \quad a_n(k) = \begin{cases} \frac{(ik)^n}{n!}, & n \geq 0 \\ 0, & n < 0. \end{cases}$$

Substituting this series into formula (2.10) gives

$$t(k) \approx t^{\text{exp}}(k) = \sum_{n=1}^{N} (\lambda_n - |n|) \frac{(-1)^n |k|^{2n}}{(n!)^2}.$$  

Although one would expect this approximation to be more accurate for small $q$, we obtained reasonable results in our examples even when $q$ was not small.

4. From $t(k)$ to $\gamma$

By [18] the $\tilde{\delta}$ inversion $t \rightarrow \gamma$ is well-posed and even contributes some smoothing. The fact that the $\tilde{\delta}$ equation must be solved independently for each $x$ in the region of interest to obtain $\gamma(x)$ suggests the use of parallelization in a numerical method. Here a 2-D adaptation of the method of product integrals presented in [1] in 1-D is used to solve the $\tilde{\delta}$ equation in parallel for the $x$ values in the region of interest. The idea of the method is to factor the integrand into its smooth part and its singular part and approximate the smooth part with a simple function, such as an interpolatory polynomial. The new integrand is then computed analytically where possible. We describe the method applied to the weakly singular second-order Fredholm integral equation (2.12) below. We mention that in the numerical solution of the $\tilde{\delta}$ equation, no radially symmetry of $\gamma$ or $t(k)$ is assumed.

For $s \in \mathbb{C} \setminus 0$ write equation (2.12) as

$$\mu(x,s) = 1 + \frac{1}{4\pi^2} \int_{\mathbb{R}^2} H(s,k)L(x,k)\mu(x,k)dk_1dk_2$$

\[4.1\]
where $k = k_1 + ik_2$ and 

$$
(4.2) \quad H(s, k) := \frac{1}{(s - k)} \quad \text{and} \quad L(x, k) := \frac{t(k)}{k} e^{-x(k)}.
$$

In [21] it is shown that for $\gamma \in C^{2+m}(\Omega)$, $m \geq 1$, $|t(k)| \leq C|k|^{-m}$ for large $|k|$. Thus, the integrand in (4.1) approaches zero as $|k| \to \infty$, and for numerical purposes we choose $A, C > 0$ sufficiently large and instead solve

$$
\mu(x, s) = 1 + \frac{1}{4\pi^2} \int_{-A}^{A} \int_{-C}^{C} H(s, k) L(x, k) \mu(x, k) dk_1 dk_2.
$$

Next, define a mesh on $[-A, A] \times [-C, C]$ in such a way that $k_1 = 0, k_2 = 0$ is not a mesh point. Let 

$$
\begin{align*}
\mathcal{u}_j & = -A + j h_u, & j = 0, \ldots, N + 1 \\
\mathcal{v}_i & = -C + i h_v, & i = 0, \ldots, N + 1.
\end{align*}
$$

where $u_{j+1} - u_j = h_u > 0, j = 0, \ldots, N$ and $v_{i+1} - v_i = h_v > 0, i = 0, \ldots, N$.

Since $\mu \sim 1$ for $|k|$ large, on the set $S := \{[\mathcal{u}_j, u_{j+1}] \times [\mathcal{v}_i, v_{i+1}] : j \in \{0, N - 1\}$ or $i \in \{0, N - 1\}$ of outer mesh elements we set $\mu \equiv 1$. Then we wish to solve

$$
(4.3) \quad \mu(x, s) = g(x, s) + \frac{1}{4\pi^2} \sum_{j=1}^{N-1} \sum_{i=1}^{N-1} \int_{\mathcal{u}_j}^{u_{j+1}} \int_{\mathcal{v}_i}^{v_{i+1}} H(s, k) L(x, k) \mu(x, k) dk_1 dk_2
$$

where

$$
(4.4) \quad g(x, s) := 1 + \frac{1}{4\pi^2} \int_S H(s, k) L(x, k) dk_1 dk_2.
$$

We then approximate the function $f(x, k) := L(x, k) \mu(x, k)$ by an interpolatory polynomial. Here, we use bilinear interpolation and introduce the notation

$$
[f(x, k_1, k_2)]_{ij} := (1 - t)(1 - w)f_{j,i} + t(1 - w)f_{j+1,i} + twf_{j+1,i+1} + (1 - t)wf_{j,i+1}
$$

where $f_{j,i} := f(x, u_j, v_i)$, $t := \frac{k_1 - u_j}{h_u}$, and $w := \frac{k_2 - v_i}{h_v}$.

For $x \in \Omega$ define the numerical integration operator by

$$
(4.5) \quad \kappa_N \mu(x, s) := \sum_{j=1}^{N-1} \sum_{i=1}^{N-1} \int_{\mathcal{u}_j}^{u_{j+1}} \int_{\mathcal{v}_i}^{v_{i+1}} H(s, k) [f(x, k_1, k_2)]_{ij} dk_1 dk_2.
$$

To obtain a linear system, we choose $s$ to be the nodes of the inner mesh elements $\{s = (u_j, v_i)\}_{i=1}^N$. Then to form $g(x, s)$ and $\kappa_N \mu(x, s)$ the following integrals must be evaluated for $j, i = 0, \ldots, N$ and $s = (u_m, v_n)$, $m, n = 1, \ldots, N$.

$$
(4.6) \quad J_{\alpha \beta}^{ij}(s) := \int_{\mathcal{u}_j}^{u_{j+1}} \int_{\mathcal{v}_i}^{v_{i+1}} \frac{k_1^\alpha k_2^\beta}{(s - k)} dk_1 dk_2, \quad \alpha, \beta \in \{0, 1\}.
$$

Note that when $s$ lies on a corner of the mesh element over which we are integrating, an integrable singularity will be present in the integrand. When $s$ does not coincide with a corner of the mesh element over which we are integrating, the above integrals are not singular, and they can be computed using a numerical quadrature method such as 2-D Gauss-Legendre quadrature. The singular integrals can be evaluated analytically using residue calculus.
Denote \( \mu_{ji}(x) := \mu(x, (u_j, v_j)) \). By regrouping terms, the numerical integration operator in (4.5) can now be written as

\[
\kappa_N \mu(x, s) = \sum_{j=1}^{N-1} \sum_{i=1}^{N-1} A^{ji}(x) \mu_{ji}(x)
\]

where

\[
A^{ji}(x) = a_{ji}(x) J_{00}^{ji}(s) + b_{ji}(x) J_{10}^{ji}(s) + c_{ji}(x) J_{01}^{ji}(s) + d_{ji}(x) J_{11}^{ji}(s).
\]

Choose \( \{s = (u_k, v_k)\}_{k=1}^{N} \) defining an \( N^2 \) by \( N^2 \) matrix \( \mathbf{A}(x) = (A^{ji}(x)) \) where \( A^{ji}(x) \) is the linear combination of the \( J_{\alpha\beta}^{ji}(u_k, v_k) \) above, \( \alpha, \beta \in \{0, 1\} \). This results in the linear system

\[
I \mu(x) - \mathbf{A} \tilde{\mu}(x) = \mathbf{g}(x)
\]

where \( I \) is the \( N^2 \) by \( N^2 \) identity matrix. This system can be solved by equating the real and imaginary parts to obtain two linear systems in real variables with two vectors of unknowns. Namely,

\[
(I - \text{Re}(\mathbf{A}))a - \text{Im}(\mathbf{A})b = \text{Re}(\mathbf{g})
\]

\[
(I + \text{Re}(\mathbf{A}))b - \text{Im}(\mathbf{A})a = \text{Im}(\mathbf{g})
\]

where \( \mu = a + ib \). Solving the linear system gives \( \{\mu(x, (u_j, v_j))\}_{j=1}^{N} \).

Note that the factors \( J_{\alpha\beta}^{ji}(s) \) in the matrix \( \mathbf{A} \) are independent of \( x \), so they need only be computed once and stored. Then in parallel, the matrix \( \mathbf{A} \) is assembled by forming the linear combination (4.8) and the resulting systems (4.9) are solved.

5. Numerical Examples

In this section we test the algorithm for three high-contrast conductivities in \( C^3(\Omega) \) and a high contrast conductivity in \( C^4(\Omega) \) which contains a dip near the boundary of \( \Omega \). The first three examples increase in magnitude from 4 to approximately 6 and 8. We compare the corresponding scattering transforms and the reconstructions. The conductivities for Examples 1, 2, and 3 are defined by the following formula. Fix \( 0 < \rho < 1 \) and let \( F_\rho \in C^3_0(\mathbb{R}) \) be given by

\[
F_\rho(x) := (x^2 - \rho^2)^4, \quad -\rho \leq x \leq \rho
\]

and \( F_\rho(x) \equiv 0 \) for \( |x| > \rho \). Let

\[
\gamma(x) := (\alpha F_\rho(|x|) + 1)^2,
\]

Then the support of \( \gamma(x) - 1 \) is the interval \([-\rho, \rho] \). 

Example 1: \( \rho = 1/4 \) and \( \alpha = 16\rho^{-6} \).

Example 2: \( \rho = 1/4 \) and \( \alpha = 23\rho^{-6} \).

Example 3: \( \rho = 1/4 \) and \( \alpha = 29\rho^{-6} \).

The plots of \( \gamma_1, \gamma_2, \) and \( \gamma_3 \) are found with the reconstructions in Figure 2. In each of these examples, 49 eigenvalues were computed using the method described in Section 3. The upper and lower bounds were in very close agreement. The approximate scattering transform \( t^{\gamma_\rho}(k) \) was then computed using the series (3.1) with \( N = 49 \) and the lower bounds for the eigenvalues. The scattering transforms \( t^{\gamma_\rho}(k) \) corresponding to \( \gamma_1, \gamma_2, \) and \( \gamma_3 \) are plotted in Figure 1. One observes that the scattering transform increases in amplitude with \( \gamma \), and the divergence of the series (3.1) becomes more marked as the amplitude of \( \gamma \) increases. Reconstructions
of $\gamma_1$, $\gamma_2$, and $\gamma_3$ were obtained by solving the $\hat{\theta}$ equation as described in Section 4. The $\hat{\theta}$ equation was solved on the $k$ mesh $[-30,30]^2$ with stepsize $h \approx 1.5$ and for the $41$ $x$ values $x = (-0.4,0),(-0.38,0),\ldots,(0.38,0),(0.4,0)$. Since
\begin{equation}
\gamma^{1/2}(x) = \lim_{k \to 0} \mu(x,k),
\end{equation}
the value of $\gamma^{1/2}(x)$ was approximated by $\Re(\mu(x,k))$ at a node $k$ with minimum norm. The $\gamma_i$ reconstructed thusly are plotted in Figure 2. The relative error in the reconstruction increases as the amplitude of $\gamma$ increases. The relative errors for $\gamma_1$, $\gamma_2$, and $\gamma_3$ were .127, .190, and .251 respectively.

Example 4: Here, the conductivity function under consideration is defined by the following formula. Fix $0 < \rho < 1$ and let $F_\rho \in C_0^1(\mathbb{R})$ be given by
\begin{equation}
F_\rho(x) := (x^2 - \rho^2)^4 \cos \frac{3\pi x}{2\rho}, \quad -\rho \leq x \leq \rho
\end{equation}
and $F_\rho(x) \equiv 0$ for $|x| > \rho$. Let
\begin{equation}
\gamma(x) := (10F_{3/4}(|x|) + 1)^2,
\end{equation}
Then the support of $\gamma(x) - 1$ is the interval $[-3/4,3/4]$. See Figure 3 for a plot of the conductivity and the corresponding potential.

Using the method described in Section 3, 75 eigenvalues of the Dirichlet-to-Neumann map were computed. Again, the upper and lower bounds were in very close agreement. The approximate scattering transform $t^{exp}(k)$ was then computed using the series (3.1) with $N = 75$. To validate the function $t^{exp}(k)$ computed from the series (3.1), the scattering transform was also computed from the definition by solving the forward problem. First, the functions $\mu_{LS}(x,k)$ were computed on the $k$ mesh $[-30,30]^2$ by solving the Lippmann-Schwinger equation
\begin{equation}
\mu - 1 = -g_k * (q\mu)
\end{equation}
using a projection method described in [21]. Recall $g_k$ is the Fadeev’s Green’s function defined by (2.9). The scattering transform was then computed by applying Gauss-Legendre quadrature to the definition using the computed values $\mu_{LS}$:
\begin{equation}
t_{LS}(k) := \int_{\mathbb{R}^2} e_k(x)\mu_{LS}(x,k)q(x)dx, \quad k \in \mathbb{C} \setminus 0.
\end{equation}
Plots of $t^{exp}(k)$, $t_{LS}(k)$, and $q(2|k|)$ are found in Figure 4. Note that except for small $|k|$ values, the function $t_{LS}(k)$ is very similar to $q(2|k|)$.

Reconstructions of $\gamma$ were obtained by solving the $\hat{\theta}$ equation using the method described in Section 4 and using both $t^{exp}(k)$ and $t_{LS}(k)$ as data in the equation. The $\hat{\theta}$ equation was solved on the uniform $k$-mesh $[-20,20]^2$ with $h = 1$. Again, the value of $\gamma^{1/2}(x)$ was approximated by $\Re(\mu(x,k))$ at a node $k$ with minimum norm. The $\hat{\theta}$ equation was solved for the $x$ values $(-0.8,0),(-0.75,0),\ldots,(0.75,0),(0.8,0)$, and the $\gamma$’s reconstructed from $t^{exp}(k)$ and $t_{LS}(k)$ are plotted in Figure 5. The relative error for $\gamma$ reconstructed from $t^{exp}$ is .316 while the relative error for $\gamma$ reconstructed from $t_{LS}$ is .082. Although the reconstructed conductivity profiles are not perfectly symmetric, they are nearly so. Note that some incorrect oscillation is present in the reconstruction from $t^{exp}$. This may be due to the divergence of the series for large $|k|$, although such oscillations were not present in reconstructions of simpler radially symmetric examples that did not contain a dip such as those above and those in [21]. Figures 6 contains a 2-D plot of the reconstructed $\gamma$ from $t^{exp}(k)$.
RECONSTRUCTION OF HIGH CONTRAST CONDUCTIVITIES

\[ t^{exp}(k) \]

\[ 0 \quad 5 \quad 10 \quad 15 \quad 20 \quad 25 \quad 30 \quad 35 \quad 40 \quad 45 \]

\[ -6 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \]

**Figure 1.** The functions \( t^{exp}(k) \) for \( \gamma_1(x) \) (dot-dashed line), \( \gamma_2(x) \) (solid line), and \( \gamma_3(x) \) (dashed line).

**Figure 2.** Cross-sectional plots of the reconstructed \( \gamma_i \) (-+) and the actual \( \gamma \) (-) for \( i = 1 \) (upper left), \( i = 2 \) (upper right), and \( i = 3 \) (bottom center).
on the $x$ mesh $[-0.6,0.6]^2$ with mesh stepsize 0.1. The functions $\Re(\mu_{exp}(x,k))$, $\Re(\mu_{LS}(x,k))$, $\Im(\mu_{exp}(x,k))$, and $\Im(\mu_{LS}(x,k))$ are found in Figures 7 and 8.

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References

Figure 4. The functions $t^{exp}(k)$ (solid line), $t^{LS}(k)$ (dashed line), and $\tilde{q}(2|k|)$ (dot-dashed line) for Example 4.

Figure 5. Cross-sectional plot of the reconstructed $\gamma$ from $t^{exp}$ (+), the reconstructed $\gamma$ from $t^{LS}$ (-o), and the actual $\gamma$ (solid line) for Example 4.


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Figure 6. Plot of the reconstructed $\gamma$ from $t^{exp}$ for Example 4.

Figure 7. The reconstructed $\mathbb{R}(\mu_{exp}(x, k))$ (left column), $\mathbb{R}(\mu_{LS}(x, k))$ (center column) and $\mathbb{R}(\mu_{LS}(x, k)) - \mathbb{R}(\mu_{exp}(x, k))$ (right column) for $x = 0, 1, 2, 4$ (top to bottom).
Figure 8. The reconstructed $\Im(\mu_{exp}(x, k))$ (left column), $\Im(\mu_{LS}(x, k))$ (center column) and $\Im(\mu_{LS}(x, k)) - \Re(\mu_{exp}(x, k))$ (right column) for $x = 0, 1, 2, 4$ (top to bottom).