Dynamic multi-source X-ray tomography using a spacetime level set method

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A novel variant of the level set method is introduced for dynamic X-ray tomography. The target is allowed to change in time while being imaged by one or several source–detector pairs at a relatively high frame-rate. The algorithmic approach is motivated by the results in [22], showing that the modified level set method can tolerate highly incomplete projection data in stationary tomography. Furthermore, defining the level set function in spacetime enforces temporal continuity in the dynamic tomography context considered here. The tomographic reconstruction is found as a minimizer of a nonlinear functional. The functional contains a regularization term penalizing the $L^2$ norms of up to $n$ derivatives of the reconstruction. The case $n = 1$ is shown to be equivalent to a convex Tikhonov problem that has a unique minimizer. For $n \geq 2$ the existence of a minimizer is proved under certain assumptions on the signal-to-noise ratio and the size of the regularization parameter. Numerical examples with both simulated and measured dynamic X-ray data are included, and the proposed method is found to yield reconstructions superior to standard methods such as FBP or non-negativity constrained Tikhonov regularization and favorably comparable to those of total variation regularization. Furthermore, the methodology can be adapted to a wide range of measurement arrangements with one or more X-ray sources.

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1. Introduction

We consider the use of X-ray tomography for imaging moving targets. The key difficulty is that while the source–detector pair of a CT device rotates between recording two projection images, the target has already changed. Periodic movement such as the beating of a heart can be satisfactorily imaged using gating [32,18]. However, problems remain with tomographic imaging of non-periodic changes such as the flow of contrast agent inside blood vessels in angiography.

This work is motivated by multi-source tomography setups such as described already in 1983 in [33]. Consider placing several X-ray sources to irradiate a moving target simultaneously, and a corresponding set of X-ray detectors with high framerate. See Fig. 2. There are no moving parts in this arrangement, reducing calibration issues and offering simplifications for engineering and manufacturing.

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The above tomographic sampling geometry is quite peculiar. Temporal resolution can be very high as detectors with framerate of 400 fps or more are available off-the-shelf. However, the number of projection directions is severely limited simply by spatial constraints in placing X-ray tubes and bulky detectors in three-dimensional space. The experimental setup in [33] featured 28 source–detector pairs, which is still very few from the tomographic resolution-theory point of view [25]. Keeping the number of source–detector low, preferably under ten, offers significant economical and engineering advantages but leads to a very ill-posed tomographic problem.

There are encouraging results in the literature of extremely sparse-angle stationary tomography [37,23,35,36,3,15,19,14], including approaches based on level set methods [40,11,30,22,39,21]. See the introduction of [14] for more references. The above studies suggest that even less than 10 projection directions can be enough for tomographic reconstruction if the incomplete measurement information is effectively complemented with a priori information.

In particular, the modified level set method approach introduced in [22] is known to suppress sparse-data and limited-angle artifacts very effectively. The modification is to model the X-ray attenuation coefficient inside the level set by the continuous level set function itself instead of a constant, as in the classical level set method [29,27,16,41,9,6,28,5,7,10,8]. See Fig. 3.

However, incorporating time in the reconstruction calls for a novel algorithmic approach. In [26], three of the present authors extended the modified level set method to the dynamic case using a spacetime approach. Temporal regularization, or the promotion of relatively slow changes of the target in time, is offered simply by the requirement that the level set function is continuous in spacetime.

The promising results in [26] are based on simulated data of a simple target only. The goal of this paper is to generalize and analyze the spacetime level-set method introduced in [26] and to test it numerically in $(2 + 1)$-dimensional cases using

1. simulated data of more complicated targets than those used in [26], and
2. measured X-ray data of a temporally changing object.

Let us describe the principle of the spacetime level set method in the $(2 + 1)$-dimensional setting. The X-ray attenuation is modelled by a function of the form $f(u)$, where $u = u(x, y, t)$ is a smooth function and $f : \mathbb{R} \to \mathbb{R}$ is given by

$$f(\tau) = \begin{cases} \tau, & \text{if } \tau \geq 0, \\ 0, & \text{if } \tau < 0. \end{cases}$$

This means that the attenuation is zero outside the level set, in other words in the set \{$(x, y, t) | u(x, y, t) < 0$\}. Further, the attenuation coincides with $u(x, y, t)$ inside the level set, or in technical terms in \{$(x, y, t) | u(x, y, t) > 0$\}.

We find the level set function $u$ as a minimizer of the functional $F_n : H^1(\Omega) \to \mathbb{R}$,

$$F_n(u) = \|A f(u) - m\|_{L^2(\Omega)}^2 + \alpha \sum_{1 \leq |\beta| \leq n} \|D^\beta u\|_{L^2(\Omega)}^2,$$

where $A$ is an operator modeling 2D Radon transforms measured at several times, $\beta = (\beta_1, \beta_2, \beta_3)$ is a multi-index with $|\beta| = \beta_1 + \beta_2 + \beta_3$, and $\alpha > 0$ is a regularization parameter.

For the special case $n = 1$ studied also in [22,26] we prove that minimizing $F_n$ is equivalent to non-negativity constrained Tikhonov regularization

$$\arg \min_{u \in H^1(\Omega)} \left\{ \|A u - m\|_{L^2(\Omega)}^2 + \alpha \|\nabla u\|_{L^2(\Omega)}^2 \right\} \geq 0,$$

which has a unique solution. This result gives new insight into the connection between level set methods and Tikhonov regularization. In particular, this explains our numerical observation that the level set function never attains very negative values when $n = 1$.

Furthermore, we analyse the proposed method in the cases $n \geq 2$. Despite the fact that the case $n = 1$ essentially reduces to a convex minimization problem, for $n \geq 2$ such result is not available and we have to solve the non-convex minimization problem $\arg \min_{u \in H^1(\Omega)} F_n(u)$. See Fig. 1 for an illustration of the non-convexity. Moreover, the functional $F_n$ is not coercive in $H^1$. However, we are able to use a non-standard argument to show that $F_n$ has at least one global minimizer. Our strategy of proof is based on requiring the regularization parameter to satisfy a bound involving the signal-to-noise ratio, which is a practically relevant quantity that can typically be estimated from measurements.

The new spacetime level-set method is defined in a general form and can be readily extended to other measurement geometries. One of the most interesting extensions is one source–detector pair imaging a moving object on a slowly rotating platform. The rotation would provide a complete collection of projection directions for a static target, but in the present context the object is allowed to change in time.

Dynamic X-ray tomography is not a new idea. Regular CT devices are used dynamically all the time: a search in a scholarly database using the search term “dynamic CT” yields more than two million hits. However, this is typically based on filtered back-projection (FBP) algorithms, requiring fine angular sampling and thus being applicable only to slow or periodic motion. While there are dedicated dynamic FBP variants [34,20] and deformation-specific approaches [12], tracking
The function $\tau \mapsto |f(\tau) - 1|^2$ plotted versus $\tau$ to illustrate the non-convexity of the functional $F_n$. 

Fig. 2. Arrangements of multiple source–detector pairs with no moving parts. Orange dots denote X-ray sources, while detectors are drawn in light blue color. Left: a three-dimensional arrangement. Right: a two-dimensional situation such as the one used in the computational examples studied in this work. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

fast movement with FBP remains challenging. The method proposed here is applicable to data collected with a CT machine and greatly reduces the angular sampling requirement, enabling the imaging of much faster targets.

The multi-source setting has been studied for dynamical cases as well [33,38,24], but again with FBP-type reconstruction methods. The present approach is completely different from all of the preceding methods.

This paper is organized as follows. In Section 2 we describe the spacetime level set method mathematically and provide new analytic results about its properties. Section 3 is devoted to computational implementation issues, and in Section 4 we describe the simulated and measured datasets we use to demonstrate our method. Section 5 presents the results of our algorithm when applied to the datasets. We discuss and conclude our findings in Section 6.

2. The spacetime level set method

As mentioned in the Introduction, the goal of the new reconstruction method is two-fold:

1. The method needs to enforce regularity in the reconstruction not only in spatial but also in temporal direction.
2. The method should produce acceptable results even with very sparse collections of projection directions.

In this section we describe the proposed method and analyze its properties.

2.1. Motivation and idea

We model the two-dimensional object of interest at time $t$ by a non-negative X-ray attenuation function $w_t(x, y) = w(x, y, t) = w(x)$. Here $w : \Omega \subset \mathbb{R}^3 \to \mathbb{R}_+$, where $\Omega = [0, N] \times [0, N] \times [0, T]$ with positive constants $N, T > 0$. We consider the tomographic measurement model

$$Aw = m,$$  
(3)
where $\mathcal{A}$ is an operator consisting of a “stack” of standard 2D Radon transforms $\mathcal{R}$,

$$(\mathcal{A}w)(L, t) := (\mathcal{R}w_t)(L) = \int_L w_t(x, y) dx dy$$

with $L = L(\theta, d) = \{(x, y) \in \mathbb{R}^2 \mid x \cos \theta + y \sin \theta = d\}$ denoting a line in the $(x, y)$-plane determined by $\theta \in [0, \pi)$ and $d \in \mathbb{R}$. The sinograms $m = m(L, t)$ at fixed times are known from the X-ray measurements.

Motivated by [22] and a spacetime interpretation of the object as shown in Fig. 4, we propose modelling the attenuation function as $w = f(\Phi_n)$ where $\Phi_n : \Omega \rightarrow \mathbb{R}$ is a minimizer of the functional $F_n : H^1(\Omega) \rightarrow \mathbb{R}$,

$$F_n(u) = \|Af(u) - m\|_{L^2(\mathcal{E})}^2 + \alpha \sum_{1 \leq |\beta| \leq N} \|D^{\beta}u\|_{L^2(\Omega)}^2.$$  \hspace{1cm} (4)

Here $E$ denotes the set of all lines in the $(x, y)$-plane at different time instants, $\beta = (\beta_1, \beta_2, \beta_3)$ is a multi-index with $|\beta| = \beta_1 + \beta_2 + \beta_3$, and $\alpha > 0$ is a regularization parameter. Note that we do not penalize the $L^2$-norm of $u$, since we would like to allow solutions that are close to segmented, piecewise constant functions.

Analogously to [22], the approach described above is motivated by level set methods. More precisely, we consider $\Phi_n : \Omega \rightarrow \mathbb{R}$ as a level set function such that

(i) the X-ray attenuation function is supported inside the level set $\{(x, y) \in \Omega : \Phi_n(x, y) = 0\}$, and
(ii) the attenuation function is given by the level function itself inside the level set and by zero elsewhere.

Compared to [22], the new approach is designed for non-stationary $(2 + 1)$-dimensional CT imaging by adding the time variable $t$ to the level set function and, moreover, it includes higher order derivatives in the regularization term. Consequently, the operator $\mathcal{A}$ here is defined as a stack of 2D Radon transforms incorporating several instants of time.

Let us fix some notation and prove a lemma regarding the operator $\mathcal{A}$. Denote $\Omega = \tilde{\Omega} \times [0, T]$ with $\tilde{\Omega} = [0, N] \times [0, N]$ and $N, T > 0$. Furthermore, write $E = \tilde{E} \times [0, T]$ where $\tilde{E}$ denotes all the lines in the $(x, y)$-plane. The mapping $\mathcal{A} : L^2(\Omega) \rightarrow L^2(\mathcal{E})$ is given by

$$(\mathcal{A}u)(L, t) := (\mathcal{R}u_t)(L) = \int_L u_t(\hat{x}) d\hat{x}, \quad (L, t) \in \tilde{E} \times [0, T],$$

where $\mathcal{R} : L^2(\tilde{\Omega}) \rightarrow L^2(\tilde{E})$ denotes the standard 2D Radon transform and $u_t(\hat{x}) = u(\hat{x}, t) = u(x)$.
Fig. 4. Illustration of the $2 + 1$-dimensional spatio-temporal interpretation of the object. Left: Six states $w_t = w_t(x, y)$ of a dynamic 2D object at time instants $t = 1, \ldots, 6$. Right: The same dynamic object considered in the 3D Euclidean $(x, y, t)$-space as $w = w(x, y, t)$.

Lemma 2.1. The operator $A : L^2(\Omega) \to L^2(E)$ satisfies
\[ \langle A(1), A(1) \rangle_{L^2(E)} \geq C_A, \]
where $C_A > 0$ is constant, and $1 : \Omega \to \mathbb{R}, 1(x) = 1$ for all $x \in \Omega$.

Proof. Assume $u, v \in L^2(\Omega)$ and compute
\[
\langle Au, Av \rangle_{L^2(E)} = \langle Au, Av \rangle_{L^2(\hat{E}^\mathbb{R}[0, T])}
= \int_0^T \int_{\hat{E}} Au(L, t) Av(L, t) dL dt
= \int_0^T \int_{\hat{E}} (Ra(u(\cdot, t))) (L) (Ra v(\cdot, t))(L) dL dt
= \int_0^T \int_{\hat{E}} (R^*Ra(u(\cdot, t))) (\hat{x}) v(\hat{x}, t) d\hat{x} dt.
\]
According to [25, Thm. 1.5, p. 15]
\[ (R^*Ra(u(\cdot, t))) (\hat{x}) = \int_{\hat{\Omega}} \frac{1}{|\hat{x} - \hat{z}|} u(\hat{z}, t) d\hat{z} \]
and thus
\[ \langle Au, Av \rangle_{L^2(\Omega)} = \int_0^T \int_{\hat{\Omega}} \left( \int_{\hat{\Omega}} \frac{1}{|\hat{x} - \hat{z}|} u(\hat{z}, t) d\hat{z} \right) v(\hat{x}, t) d\hat{x} dt. \]
Now choose $u = v = 1$ to get
\[ \langle A(1), A(1) \rangle_{L^2(E)} = \int_0^T \int_{\hat{\Omega}} \int_{\hat{\Omega}} \frac{1}{|\hat{x} - \hat{z}|} d\hat{z} d\hat{x} dt \geq \frac{1}{\text{diam}(\hat{\Omega})^2 T} \]
\[ =: C_A, \]
where $\text{diam}(\hat{\Omega}) = \max(|\hat{x}_1 - \hat{x}_2| : \hat{x}_1, \hat{x}_2 \in \hat{\Omega})$. □
2.2. Case \( n = 1 \) equivalent to Tikhonov regularization

In the following we shall show that minimizing \( F_1 \) is equivalent to the more classical non-negativity constrained Tikhonov problem minimizing the functional \( K : H^1(\Omega) \to \mathbb{R} \),

\[
K(u) := \left\| Au - m \right\|^2_{L^2(\Omega)} + \alpha \| \nabla u \|^2_{L^2(\Omega)}
\]

subject to \( u \geq 0 \). We begin by showing that \( K : H^1(\Omega) \to \mathbb{R} \) is lower semicontinuous (l.s.c.) with respect to the weak topology of \( H^1(\Omega) \). For this it suffices to show that \( \tilde{K} : H^1(\Omega) \to \mathbb{R} \),

\[
\tilde{K}(u) = \| Au - m \|^2_{L^2(\Omega)}
\]

is weakly lower semicontinuous (it is well-known that the regularization part \( u \mapsto \| \nabla u \|^2_{L^2(\Omega)} \) is weakly lower semicontinuous in \( H^1(\Omega) \)).

**Theorem 2.2.** The functional \( \tilde{K} : H^1(\Omega) \to \mathbb{R} \) defined above is weakly lower semicontinuous.

**Proof.** As the embedding from \( H^1(\Omega) \) to \( L^2(\Omega) \) is compact, it suffices to show that \( \tilde{K} : L^2(\Omega) \to \mathbb{R} \) is continuous with respect to the strong topology of \( L^2(\Omega) \).

The standard 2D Radon transform \( R : L^2(\tilde{\Omega}) \to L^2(\hat{E}) \) is linear and continuous [25] i.e. it is bounded,

\[
\| Ru_t(L) \|_{L^2(\hat{E})} \leq \| R \| \| u_t \|_{L^2(\Omega)}, \quad u_t(\hat{x}) = u(\hat{x}, t) = u(x),
\]

and hence

\[
\| Au \|_{L^2(\hat{E})} = \left( \int_{\hat{E}} |Au(L)|^2 dL \right)^{1/2}
\]

\[
= \left( \int_0^T \int_{\hat{E}} |Ru_t(L)|^2 dL dt \right)^{1/2}
\]

\[
\leq \| R \| \left( \int_0^T \int_{\Omega} |u_t(\hat{x})|^2 d\hat{x} dt \right)^{1/2}
\]

\[
= \| R \| \| u \|_{L^2(\Omega)}.
\]

In other words, \( A : L^2(\Omega) \to L^2(\hat{E}) \) is a bounded linear operator, i.e. it is continuous. Moreover, the mapping \( m \mapsto \| u - m \|^2_{L^2(\Omega)} \) from \( L^2(\hat{E}) \) to \( \mathbb{R} \) is continuous and thus \( \tilde{K} : L^2(\Omega) \to \mathbb{R} \) is continuous w.r.t. the strong topology of \( L^2(\Omega) \).

This implies the lower semicontinuity of \( K \) in \( H^1(\Omega) \). \( \Box \)

**Theorem 2.3.** The functional \( K : H^1(\Omega) \to \mathbb{R} \) is coercive.

**Proof.** We prove the coercivity of \( K \) by showing that for any \( M > 0 \) the set \( \{ u \in H^1(\Omega) : K(u) \leq M \} \) is bounded. Assume \( u \in H^1(\Omega) \) satisfies \( K(u) \leq M \) with some \( M > 0 \) and write

\[
u = u + r \mathbf{1},
\]

where

\[
\int_\Omega w(x) dx = 0, \quad r = \frac{1}{|\Omega|} \int_\Omega u(x) dx.
\]

Then

\[
\| \nabla w \|^2_{L^2(\Omega)} = \| \nabla u \|^2_{L^2(\Omega)} \leq \frac{K(u)}{\alpha} \leq \frac{M}{\alpha}.
\]

Moreover, the Poincaré inequality gives the estimate

\[
\| w \|_{L^2(\Omega)} \leq C_P \| \nabla w \|_{L^2(\Omega)} \leq C_P \sqrt{\frac{M}{\alpha}}.
\]
with some constant $C_p > 0$, and hence

$$\|w\|_{H^1(\Omega)} \leq \sqrt{(C_p^2 + 1) \frac{M}{\alpha}}.$$  

Compute then

$$\|A(w + r1) - m\|_{L^2(E)}^2 = r^2 \langle A1, A1 \rangle_{L^2(E)} + 2r \langle A1, Aw - m \rangle_{L^2(E)}$$

$$+ \langle Aw, Aw \rangle_{L^2(E)} - 2 \langle Aw, m \rangle_{L^2(E)} + \|m\|^2_{L^2(E)}$$

$$= r^2 \langle A1, A1 \rangle_{L^2(E)} + O(r),$$

where the last equality follows from the Cauchy–Schwarz inequality and the fact that $A$ is a bounded operator. Thus, since $\langle A1, A1 \rangle_{L^2(E)} \geq C_A > 0$ by Lemma 2.1, there exists $C_r > 0$ such that $\|A(w + r1) - m\|_{L^2(E)} > M$ for $r > C_r$. Hence our assumption implies $r \leq C_r$.

Combining the two estimates we have

$$\|u\|_{H^1(\Omega)} \leq \|w\|_{H^1(\Omega)} + r|\Omega| \leq \sqrt{(C_p^2 + 1) \frac{M}{\alpha}} + C_r|\Omega|,$$

i.e. the set $\{u \in H^1(\Omega) : K(u) \leq M\}$ is bounded. □

Define the indicator function $I_X : H^1(\Omega) \to \mathbb{R}$ of the set $X = \{u \in H^1(\Omega) : u \geq 0\}$ by

$$I_X(u) = \begin{cases} 0, & \text{if } u \in X, \\ +\infty, & \text{if } u \in H^1(\Omega) \setminus X. \end{cases}$$

This is a convex function and moreover it is weakly lower semicontinuous since the set $X$ is closed in weak topology.

**Proposition 2.4.** The minimization problem

$$\arg \min_{u \in H^1(\Omega)} \{K(u) : u \geq 0\} = \left\{ \frac{\|Au - m\|^2_{L^2(E)}}{2} + \alpha \|\nabla u\|^2_{L^2(\Omega)} \right\},$$

(5)

has a unique minimizer.

**Proof.** The proof is quite standard but we present it for the convenience of the reader. First we note that the functional $K$ is strictly convex; take $u_1, u_2 \in H^1(\Omega), u_1 \neq u_2$, compute

$$\frac{\partial}{\partial \tau^2} K(\tau u_1 + (1 - \tau)u_2) = 2\|A(u_1 - u_2)\|^2_{L^2(E)} + \|\nabla(u_1 - u_2)\|^2_{L^2(\Omega)},$$

and note that this quantity is strictly positive for each $\tau \in (0, 1)$. Namely, if $\nabla(u_1 - u_2) \neq 0$, then the claim is obvious, and if $\nabla(u_1 - u_2) = 0$, then $u_1 - u_2$ is constant which by Lemma 2.1 implies $A(u_1 - u_2) \neq 0$ and the claim is obvious again.

Now the constrained minimization problem (5) is equivalent to the unconstrained strictly convex minimization problem

$$\arg \min_{u \in H^1(\Omega)} \{K(u) + I_X(u)\}.$$  

By [1, Theorem 3.3.4] there exists a minimizer for this problem and the strict convexity guarantees that the minimizer is unique. □

Now we are ready to verify a connection between the minimization of $F_1$ and the non-negativity constrained Tikhonov problem.

**Theorem 2.5.** Assume that

$$\arg \min_{u \in H^1(\Omega)} \{K(u) : u \geq 0\} = \{w\}.$$  

Then we have the following:

1. If $w \neq 0$, then

$$\arg \min_{u \in H^1(\Omega)} F_1(u) = \{w\}.$$
2. If \( w = 0 \), then
\[
\arg \min_{u \in H^1(\Omega)} F_1(u) = \{ r \mathbf{1} : r \leq 0 \}.
\]

**Proof.** Take any \( u \in H^1(\Omega) \) and set
\[
u = u^+ + u^-,
\]
where \( u^+ = \max(u, 0) \) and \( u^- = \min(u, 0) \). Then \( u^+, u^- \in H^1(\Omega) \) (see e.g. [17]), and we have
\[
F_1(u) = \| Af(u) - m \|^2_{L^2(\Omega)} + \alpha \| \nabla u \|^2_{L^2(\Omega)}
\]
\[
= \| Au^+ - m \|^2_{L^2(\Omega)} + \alpha \| \nabla u^+ + \nabla u^- \|^2_{L^2(\Omega)}
\]
\[
= \| Au^+ - m \|^2_{L^2(\Omega)} + \alpha \| \nabla u^+ \|^2_{L^2(\Omega)} + \alpha \| \nabla u^- \|^2_{L^2(\Omega)}.
\]

By the assumption we know that the term \( \| Au^+ - m \|^2_{L^2(\Omega)} + \alpha \| \nabla u^+ \|^2_{L^2(\Omega)} \) is minimized by the choice \( u^+ = w \), while the last term \( \alpha \| \nabla u^- \|^2_{L^2(\Omega)} \) is minimized by the choice \( u^- = r \mathbf{1} \) with some \( r \leq 0 \). However, in the case \( u^+ = w \neq 0 \) we need to have \( u^- = 0 \). \( \square \)

2.3. Case \( n \geq 2 \): Existence of a global minimizer

In this section we prove that the functional \( F_n : H^n(\Omega) \to \mathbb{R} \)
\[
F_n(u) = \| Af(u) - m \|^2_{L^2(\Omega)} + \alpha \sum_{1 \leq |\beta| \leq n} \| D^\beta u \|^2_{L^2(\Omega)}
\]
has a global minimizer for \( n \in \{2, 3, \ldots\} \). The functional \( F_n \) is not coercive in \( H^n \), which can be seen by choosing \( u = r \mathbf{1} \) and letting \( r \to -\infty \) in which case \( F_n(u) \to \| m \|^2_{L^2(\Omega)} \) while \( \| u \|_{H^n(\Omega)} \to \infty \). Hence, to establish the existence of a global minimizer we employ the following facts instead of the coercivity argument:

(i) the global minimizer(s) of \( F_n \) must lie in a bounded ball of \( H^n(\Omega) \), that is, there exists a finite constant \( R > 0 \) such that for some \( \xi \in (0, 1) \)
\[
\left\{ v \in H^n(\Omega) : F_n(v) \leq \inf_{u \in H^n(\Omega)} F_n(u) + \xi \right\} \subset B(R),
\]
where \( B(R) = \{ u \in H^n(\Omega) : \| u \|_{H^n(\Omega)} \leq R \} \).

(ii) any bounded ball of \( H^n(\Omega) \) is compact with respect to the weak topology of \( H^n(\Omega) \), and

(iii) \( F_n \) is lower semicontinuous (l.s.c.) with respect to the weak topology of \( H^n(\Omega) \).

We remark that the second claim is a direct consequence of the Banach–Alaoglu theorem, see e.g. [31, Theorem 6.64]. Having proved the claims (i) and (iii), the existence of a global minimizer follows since a lower semicontinuous function attains its minimum in a compact set. To prove (i) we need to make an additional assumption on the regularization parameter because of the nonlinearity caused by the function \( f \).

The lower semicontinuity of \( F_n \) is proved essentially in the same way as in the previous subsection.

**Theorem 2.6.** The functional \( F_n : H^n(\Omega) \to \mathbb{R} \) defined above is weakly lower semicontinuous.

**Proof.** It is well-known that the regularization part \( u \mapsto \sum_{1 \leq |\beta| \leq n} \| D^\beta u \|^2_{L^2(\Omega)} \) is weakly l.s.c. so it remains the prove the lower semicontinuity of \( u \mapsto \| Af(u) - m \|^2_{L^2(\Omega)} \). The proof of this fact is essentially the same as that of **Theorem 2.2** but adding the fact that the function \( f : L^2(\Omega) \to L^2(\Omega) \) is continuous:
\[
\| f(u_1) - f(u_2) \|_{L^2(\Omega)} = \left( \int_\Omega | f(u_1(x)) - f(u_2(x)) |^2 \, dx \right)^{1/2}
\]
\[
\leq \left( \int_\Omega | u_1(x) - u_2(x) |^2 \, dx \right)^{1/2}
\]
\[
= \| u_1 - u_2 \|_{L^2(\Omega)}
\]
for any \( u_1, u_2 \in L^2(\Omega) \). \( \square \)
To conclude the existence proof we establish (i). This is done under an assumption regarding the size of the regularization parameter $\alpha > 0$ and the signal-to-noise ratio in the measurement.

**Theorem 2.7.** Assume that

(A1) the zero function is not a global minimizer of $F_n$,

(A2) the signal-to-noise ratio in the measurement is $M > 2$, i.e. assume the true model is $u^* > 0$,

$$m^* = Au^*,$$

and the measured data is $m = m^* + \epsilon$, where the noise term $\epsilon$ satisfies

$$\|\epsilon\|_{L^2(E)} \leq \frac{1}{M} \|m^*\|_{L^2(E)}.$$

(A3) the regularization parameter $\alpha \in (0, \alpha_0)$, where $\alpha_0 = \alpha_0(u^*, m^*, M)$ satisfies

$$\frac{M - 2}{M} \|m^*\|_{L^2(E)}^2 = \alpha_0(u^*, m^*, M) \sum_{1 \leq |\beta| \leq n} \|D^\beta u^*\|_{L^2(\Omega)}^2.$$

Then there exists a finite constant $R > 0$ such that

$$X(\xi) := \left\{ v \in H^0(\Omega) : F_n(v) \leq \inf_{u \in H^0(\Omega)} F_n(u) + \xi \right\} \subset B(R)$$

for some $\xi \in (0, 1)$. Moreover, there exists $v_0 \in H^0(\Omega)$ such that $F_n(v_0) = \inf_{u \in H^0(\Omega)} F_n(u)$.

**Proof.** Denote $I = \inf_{u \in H^0(\Omega)} F_n(u)$. Choose $C < \frac{M - 2}{M}$ such that

$$C \|m^*\|_{L^2(E)}^2 \geq \alpha \sum_{1 \leq |\beta| \leq n} \|D^\beta u^*\|_{L^2(\Omega)}^2$$

and $\xi > 0$ so small that

$$0 \notin X(\xi) \quad \text{and} \quad \xi < (1 - C_M) \|m\|_{L^2(E)}^2,$$

where $C_M = \frac{CM^2 + 1}{(M - 1)^2} < 1$. This is possible by the assumption that the zero function is not a minimizer of $F_n$.

Assume $u_0 \in X(\xi)$ and write it in the form

$$u_0(x) = w_0(x) + r1(x),$$

where $1(x) = 1$ for all $x \in \Omega$ and

$$\int_\Omega w_0(x) dx = 0, \quad r = \frac{1}{|\Omega|} \int_\Omega u_0(x) dx$$

with $|\Omega|$ denoting the volume of $\Omega$. Then $w_0, r1 \in H^0(\Omega)$ and

$$\|u_0\|_{H^0(\Omega)} \leq \|w_0\|_{H^0(\Omega)} + |r||\Omega|.$$  

We shall show that both of the terms on the right side are necessarily bounded by constants that do not depend on $u_0$. Let us divide the rest of the proof into three steps.

1. **Bound for $\|w_0\|_{H^0(\Omega)}$.** First note that

$$\sum_{1 \leq |\beta| \leq n} \|D^\beta w_0\|_{L^2(\Omega)}^2 = \sum_{1 \leq |\beta| \leq n} \|D^\beta u_0\|_{L^2(\Omega)}^2 \leq \frac{F_n(u_0)}{\alpha} \leq \frac{I + \xi}{\alpha}. \quad (7)$$

In addition, by the Poincaré inequality there exists a constant $C_P > 0$ such that

$$\|w_0\|_{L^2(\Omega)}^2 = \|u_0 - r1\|_{L^2(\Omega)}^2 \leq C_P \|\nabla u_0\|_{L^2(\Omega)}^2.$$

Hence,

$$\|w_0\|_{L^2(\Omega)}^2 \leq C_P \sum_{1 \leq |\beta| \leq n} \|D^\beta u_0\|_{L^2(\Omega)}^2 \leq C_P \frac{l + \xi}{\alpha}$$

and
\[ \| w_0 \|_{H^\alpha(\Omega)} \leq \left( C_p \frac{I + \xi}{\alpha} + \frac{I + \xi}{\alpha} \right)^{1/2} =: C_w. \]

2. **Bound for |r| in the case** \( r \geq 0 \). Compute

\[
\| A f (w_0 + r \mathbf{1}) - m \|_{L^2(E)}^2 = \langle A^* A f (w_0 + r \mathbf{1}), f(w_0 + r \mathbf{1}) \rangle_{L^2(\Omega)} - 2 \langle f(w_0 + r \mathbf{1}), A^* m \rangle_{L^2(\Omega)} + \| m \|_{L^2(E)}^2
= \langle A^* A f (w_0 + r \mathbf{1}), f(w_0 + r \mathbf{1}) \rangle_{L^2(\Omega)} + O(r).
\]

In order to study the first term here, set
\[
h_r = f(w_0 + r \mathbf{1}) - f(r \mathbf{1})
\]

and note that then
\[
\| h_r \|_{L^2(\Omega)} \leq \| w_0 \|_{L^2(\Omega)}. \tag{8}
\]

Now
\[
\langle A^* A f (w_0 + r \mathbf{1}), f(w_0 + r \mathbf{1}) \rangle_{L^2(\Omega)} = \langle A^* f (r \mathbf{1}), f(r \mathbf{1}) \rangle_{L^2(\Omega)} + \langle A^* A h_r, f(r \mathbf{1}) \rangle_{L^2(\Omega)} + \langle A^* A f (w_0 + r \mathbf{1}), h_r \rangle_{L^2(\Omega)} + \langle A^* h_r, h_r \rangle_{L^2(\Omega)}
= r^2 \langle A^* A (\mathbf{1}), \mathbf{1} \rangle_{L^2(\Omega)} + O(r)
\]
by the Cauchy–Schwarz inequality, estimate (8) and the boundedness of the linear map \( A^* A \). Thus
\[
\| A f (w_0 + r \mathbf{1}) - m \|_{L^2(E)}^2 = r^2 \langle A (\mathbf{1}), A (\mathbf{1}) \rangle_{L^2(E)} + O(r),
\]
and we know by Lemma 2.1 that \( \langle A (\mathbf{1}), A (\mathbf{1}) \rangle_{L^2(E)} \geq C_A > 0 \), and there exists \( U > 0 \) such that \( \| A f(w_0 + r \mathbf{1}) - m \|_{L^2(E)}^2 > I + \xi \) for all \( r > U \). Thus we need to have \( r \leq U \).

3. **Bound for |r| in the case** \( r < 0 \). Here we need the assumption on the regularization parameter and the signal-to-noise ratio. Set \( p = -r \), i.e.
\[
u_0 = w_0 - p \mathbf{1}, \quad p > 0.
\]

Since \( H^\alpha(\Omega) \subset H^1(\Omega) \subset L^6(\Omega) \), we have
\[
\| w_0 \|_{H^\alpha(\Omega)} \leq C_0
\]

with some \( C_0 > 0 \). Since \( u_0 \neq 0 \), we know that there exists \( a > 0 \) such that
\[
|\{ \mathbf{x} \in \Omega : w_0(\mathbf{x}) > p \}| = a
\]
and thus
\[
(a p^6)^{1/6} \leq \left( \int_{\Omega} |w_0(\mathbf{x})|^6 d\mathbf{x} \right)^{1/6} \leq C_0.
\]

that is,
\[
a \leq \frac{C_0^6}{p^6}.
\]

Using Hölder’s inequality and the fact that \( A : L^2(\Omega) \to L^2(E) \) is a bounded operator \([25]\), we obtain
\[
\| A f(w_0 - p \mathbf{1}) \|_{L^2(E)}^2 \leq C_1 \| f(w_0 - p \mathbf{1}) \|_{L^2(\Omega)}^2
\leq C_1 \int_{\Omega} \chi_{\Omega(p)}(\mathbf{x})(w_0(\mathbf{x}) - p)^2 d\mathbf{x}
\leq C_1 \| \chi_{\Omega(p)} \|_{L^2(\Omega)} \| (w_0 - p \mathbf{1}) \|_{L^2(\Omega)}
\leq C_1 a^{2/3} (\| w_0 \|_{L^6(\Omega)} + \| p \mathbf{1} \|_{L^6(\Omega)})^2
\leq C_1 \left( \frac{C_8}{p^6} \right)^{2/3} (c_7 + c_{11} p)^2
\leq C_2 p^{-2}, \tag{9}
\]
where $\Omega(p) = \{x \in \Omega : w_0(x) > p\}$. This inequality and the assumptions (A2) and (A3) together imply that

$$|r| = p \leq \frac{\sqrt{C_2}}{1 - \sqrt{C_2}} \|m\|_{L^2(E)} =: \nu,$$

(10)

where $C_2 = C_M + \left(\frac{\xi}{\|m\|^2_{L^2(E)}}\right) < 1$. To see this note that

$$F_n(u^*) = \|Au^* - m\|^2_{L^2(E)} + \alpha \sum_{1 \leq |\beta| \leq \beta_0} \|D^\beta u^*\|^2_{L^2(\Omega)} \leq \|\epsilon\|^2_{L^2(E)} + C\|m^*\|^2_{L^2(E)}$$

$$\leq \frac{1}{M^2}\|m\|^2_{L^2(E)} + C\|m^*\|^2_{L^2(E)} \leq \frac{CM^2 + 1}{(M - 1)^2}\|m\|^2_{L^2(E)} = CM\|m\|^2_{L^2(E)}$$

and since $F_n(u_0) \leq F_n(u^*) + \xi$ we have

$$\|Af(u_0) - m\|_{L^2(E)} \leq \left(\frac{CM + \frac{\xi}{\|m\|^2_{L^2(E)}}\|m\|_{L^2(E)}\right)^{1/2},$$

which, using the reverse triangle inequality

$$\|m\|_{L^2(E)} - \|Af(u_0)\|_{L^2(E)} \leq \|Af(u_0) - m\|_{L^2(E)}$$

and inequality (9), gives (10).

Putting all the three steps together, we have shown that

$$\|u_0\|_{H^1(\Omega)} \leq \|w_0\|_{H^1(\Omega)} + r |\Omega| \leq C_W + \max(U, V)|\Omega|,$$

i.e. $u_0 \in B(R)$, where $R = C_w + \max(U, V)|\Omega|$. This shows that $F_n$ has a global minimizer $v_0$. \qed

3. Computational methods

In this section we describe numerical methods for finding the minimizer of the space–time level set functional $F_n$ in the cases $n = 1$ and $n = 2$. In addition, we describe two comparison methods used for illustrating the performance of the proposed method: non-negativity constrained Tikhonov regularization and non-negativity constrained total variation regularization, both without enforcing regularity in temporal direction but simply computing a 2D reconstruction from the sparse-angle sinogram taken at the given instant.

3.1. Discretization

We discretize the spatial domain $\hat{\Omega} = [0, N] \times [0, N]$ into $N \times N$ points (pixels)

$$x_i = i, \ y_j = j, \ i, j = 1, \ldots, N,$$

and assume that the 2D Radon data, i.e. sinograms, are available at time instants

$$t_k = \frac{(k - 1) T}{N_t - 1}, \ k = 1, \ldots, N_t.$$

We approximate the function $u = u(x, y, t)$ at the point $(x_i, y_j, t_k)$ by the number $u(i, j, k)$ and organize these values into a vector $u \in \mathbb{R}^{N^2N_t}$. Finally, we assume that we have routines computing approximately the operations $Au$ and $A^Tm$ in the discretized case, i.e. the routines essentially compute the matrix–vector products $Au$ and $A^Tm$, $A \in \mathbb{R}^{(N^2N_t) \times N_m}$, where $N_m$ is the dimension of the (discrete) measurement vector $m$ and the discrete model reads as $Au = m$. 
3.2. Numerical minimization in the case \( n = 1 \)

As was shown by Theorem 2.5, the proposed level set approach in the case \( n = 1 \) is equivalent to the Tikhonov problem

\[
\arg\min_{u \in H^1_0(\Omega)} \left\{ \|Au - m\|^2_{L^2(\Omega)} + \alpha \|\nabla u\|^2_{L^2(\Omega)} \right\}.
\]

We minimize this numerically using a projected version of the Barzilai–Borwein gradient algorithm [2] and the discretized model \( Au = m \) described above, i.e. we minimize

\[
\min_{u \geq 0} \tilde{F}_1(u) := \|Au - m\|^2_2 + \alpha \|Du\|^2_2,
\]

where the elements of vector \( Du \in \mathbb{R}^{3N} \) approximate the partial derivatives of \( u \) with respect to \( x, y \) and \( t \) by the forward difference approximations

\[
\begin{align*}
\partial_x u(x_i, y_j, t_k) & \approx u(i + 1, j, k) - u(i, j, k), \\
\partial_y u(x_i, y_j, t_k) & \approx u(i, j + 1, k) - u(i, j, k), \\
\partial_t u(x_i, y_j, t_k) & \approx \frac{u(i, j, k + 1) - u(i, j, k)}{h_t}.
\end{align*}
\]

Here \( h_t = T/N_t \) is the spacing in temporal direction and it determines the amount of regularization in temporal direction; the larger the spacing \( h_t \), the less regularization in temporal direction and vice versa.

We start the projected Barzilai–Borwein algorithm from the initial guess \( u^0 = 0 \), choose the first step size to be \( \lambda_0 = 0.0001 \) and iterate as

\[
u^{\ell+1} = P \left( u^{\ell} - \lambda_\ell \nabla \tilde{F}_1(u^{\ell}) \right), \quad \ell = 0, 1, 2, \ldots,
\]

where the step size \( \lambda_\ell \) for \( \ell = 1, 2, \ldots \) is given by

\[
\lambda_\ell = \frac{(u^{\ell} - u^{\ell-1})^T(u^{\ell} - u^{\ell-1})}{(u^{\ell} - u^{\ell-1})^T(\nabla \tilde{F}_1(u^{\ell}) - \nabla \tilde{F}_1(u^{\ell-1}))},
\]

and the projection operator \( P : \mathbb{R}^l \rightarrow \mathbb{R}^l \) is defined as

\[
(P(z))_i = \max(0, z_i), \quad z = (z_1, z_2, \ldots, z_l).
\]

The gradient \( \nabla \tilde{F}_1(u) \) is computed as

\[
\nabla \tilde{F}_1(u) = 2A^T Au - 2A^T m + \alpha \nabla(\|Du\|^2_2),
\]

where \( \nabla(\|Du\|^2_2) \) consists of terms of the form

\[
\partial_{u(i,j,k)}(\|Du\|^2_2) = \left(8 + \frac{4}{h_t} \right)u(i, j, k) - 2[u(i + 1, j, k) + u(i, j + 1, k)]
\]
\[
+ u(i - 1, j, k) + u(i, j - 1, k))
\]
\[
- \frac{2}{h_t}(u(i, j, k + 1) + u(i, j, k - 1)).
\]

Here we apply zero boundary condition on the boundary of \( \Omega \).

3.3. Numerical minimization in the case \( n = 2 \)

Here we aim to minimize the functional \( F_2 \) but drop, for simplicity, the mixed derivatives from the functional and minimize

\[
\|Af(u) - m\|^2_{L^2(\Omega)} + \alpha \left(\|\nabla u\|^2_{L^2(\Omega)} + \|\Delta^2 u\|^2_{L^2(\Omega)} + \|\Delta^2 u\|^2_{L^2(\Omega)} + \|\Delta^2 u\|^2_{L^2(\Omega)} \right).
\]

Since this functional is non-differentiable due to the singularity of \( f \) at zero, we smooth-out the singularity using the approximation \( f_\delta : \mathbb{R} \rightarrow \mathbb{R} \),

\[
f_\delta(\tau) = \begin{cases} \sqrt{\tau^2 + \delta}, & \text{if } \tau > 0, \\ 0, & \text{if } \tau \leq 0. \end{cases}
\]

before applying the Barzilai–Borwein gradient algorithm for minimization. We use \( \delta = 10^{-2} \) in numerical computations. The resulting discretized functional to be minimized is
where $A$ and $Du$ are as described above and the elements in the vectors $D_xu, D_yu, D_tu \in \mathbb{R}^{N^2N_t}$ approximate the second derivatives of $u$ with respect to $x, y$ and $t$ using the central difference approximations

\[
\begin{align*}
\partial^2_x u(x_t, y_j, t_k) &\approx u(x_t + 1, y_j, k) - u(x_t, y_j, k), \\
\partial^2_y u(x_t, y_j, t_k) &\approx u(x_t, y_j + 1, k) - u(x_t, y_j, k), \\
\partial^2_t u(x_t, y_j, t_k) &\approx \frac{u(x_t, y_j, k + 1) - u(x_t, y_j, k) - u(x_t, y_j, k - 1)}{h^2_t}.
\end{align*}
\]

We remark again here that the spacing $h_t$ adjusts the amount of regularization in temporal direction. We start the Barzilai–Borwein algorithm from the initial guess $u^0 = 0$, choose the first step size to be $\lambda_0 = 0.0001$ and iterate as

\[
u^{t+1} = \nu^t - \lambda_t \nabla \tilde{F}_2(\nu^t), \quad t = 0, 1, 2, \ldots,
\]

where the step size is computed according to (11) with $\tilde{F}_1$ replaced by $\tilde{F}_2$. The gradient $\nabla \tilde{F}_2(\nu^t)$ is of the form

\[
\nabla \tilde{F}_2(\nu) = 2Df_0(\nu)A^T Af_0(\nu) - 2Df_0(\nu)A^T m + \alpha \nabla \left( \|D\nu\|^2_2 + \|D_x\nu\|^2_2 + \|D_y\nu\|^2_2 + \|D_t\nu\|^2_2 \right),
\]

where $Df_0(\nu) \in \mathbb{R}^{(N^2N_t) \times (N^2N_t)}$ is the (diagonal) Jacobian matrix of $f_0$ with diagonal elements $f'_0(u_1), f'_0(u_2), \ldots, f'_0(u_{N^2N_t})$, $\nu = (u_1, u_2, \ldots, u_{N^2N_t})$, and $\nabla \left( \|D\nu\|^2_2 + \|D_x\nu\|^2_2 + \|D_y\nu\|^2_2 + \|D_t\nu\|^2_2 \right)$ consists of terms of the form

\[
\begin{align*}
\left(32 + \frac{4}{h^2_t} + \frac{12}{h^2_t} \right) u(i, j, k) &\quad - 10 \left[ u(i + 1, j, k) + u(i, j + 1, k) + u(i - 1, j, k) + u(i, j - 1, k) \right] \\
&\quad + 2 \left[ u(i + 2, j, k) + u(i, j + 2, k) + u(i - 2, j, k) + u(i, j - 2, k) \right] \\
&\quad - \left( \frac{2}{h^2_t} + \frac{8}{h^2_t} \right) u(i, j, k + 1) + u(i, j, k - 1) \right) \\
&\quad + \frac{2}{h^2_t} u(i, j, k + 2) + u(i, j, k - 2)).
\end{align*}
\]

Here we apply the negative boundary condition $u = -1$ on the boundary of $\Omega$ rather than the zero condition, since we would ideally like to have the level set function $u$ to be negative outside the level set $\{(x, y, t) : u(x, y, t) > 0\}$. Having computed an approximation $\nu^K$ for the minimizer of $\tilde{F}_2$, we consider its projection $f(\nu^K)$ to the non-negative quadrant as the reconstruction.

### 3.4. 2D non-negativity constrained Tikhonov reconstruction

In this method we compute the 2D reconstruction at each instant as the minimizer

\[
\arg\min_{v \in H^1(\tilde{\Omega}), \nu \geq 0} \left\{ \|R v - \tilde{m}\|^2_{L^2(\tilde{\Omega})} + \alpha \|\nabla v\|^2_{L^2(\tilde{\Omega})} \right\},
\]

where $R$ is the standard 2D Radon transform and $\tilde{m}$ is the sinogram measured at the given time. We use the same discretization and forward approximation for (spatial) derivatives as described above, and solve the resulting minimization problem using projected Barzilai–Borwein analogously to Section 3.2. On the boundary of $\tilde{\Omega}$ we apply zero boundary condition.

### 3.5. 2D non-negativity constrained total variation reconstruction

Here we aim to find the minimizer

\[
\arg\min_{v \in H^1(\tilde{\Omega}), \nu \geq 0} \left\{ \|R v - \tilde{m}\|^2_{L^2(\tilde{\Omega})} + \alpha TV(v) \right\},
\]

where $TV(v)$ denotes the total variation (TV) of $v$. Again we use the same discretization as described above and, similarly e.g. to [13], we smooth-out the non-differentiability of the absolute value function in the TV term and apply the projected Barzilai–Borwein method for the resulting minimization problem. For the smoothing parameter $\gamma > 0$ in the differentiable approximation $|\tau|_\gamma := \sqrt{\tau^2 + \gamma}$, $\tau \in \mathbb{R}$, of the absolute value function we use $\gamma = 10^{-2}$. On the boundary of $\tilde{\Omega}$ we apply zero boundary condition.
4. Tomographic data

We use three different tomographic data sets for numerical tests. Two of the sets consist of simulated data and one exploits real X-ray projection data.

4.1. Simulated examples

Our first simulated example is a 2D phantom consisting of a disc that separates into two discs, one of which separates again into two discs, see the left side phantom in Fig. 5. We simulate tomographic fan-beam test data (sinograms) for this phantom at 100 time steps. The spatial resolution in the simulation is 100 × 100, i.e. the spatio-temporal (x, y, t) resolution is 100 × 100 × 100. The number of projection images taken at a single time instant is seven (7), leading to an angular step of $\frac{360}{7}$ degrees. The measurement geometry is the same for all time steps. To demonstrate random errors in the data, we add 5% Gaussian random noise

$$0.05 \cdot \max(m) \cdot X$$

to each discretely sampled component of the measurement m. In other words, the signal-to-noise ratio is 20. Here $X \sim \mathcal{N}(0, 1)$.

The second simulated test case is otherwise the same as the first one described above but the phantom being now the one shown on the right side in Fig. 5.

4.2. Real X-ray data

Let us start by discussing two challenges in producing a measured dataset for testing the spacetime level set method.

First challenge: the spacetime level set method is designed with a multisource arrangement in mind (although the method is general enough to cover other cases as well). That’s why we wanted projection data of a temporally changing object collected from several directions simultaneously. However, building a multisource device is expensive. Furthermore, it would be nice to have some tomographic experience before building a complicated multisource device to help in deciding on the design specifications (how many sources, which directions, what framerate, and so on). Additionally, our research team does have a state-of-the-art computer-controlled single-source X-ray system at its disposal.

So we decided to build a target that can be modified but remains stationary between the modifications. This way we could follow this procedure, starting with $j = 1$:

(i) Rotate the target $z^j$ in our single-source X-ray imaging system to collect tomographic data (sinogram) $R^j$.
(ii) Modify target $z^j$ slightly and call the result $z^{j+1}$.
(iii) While $j < J$, set $j := j + 1$ and go to (i).

The above process is similar in nature to “stop motion animation” used in the film industry.

Second challenge: how to choose a suitable modifiable target for our “stop motion X-ray tomography”? Inspired by claymation, or clay animation, we tried to use the modeling compound known as play-doh. (That’s the stuff children often use in their art and craft projects.) However, it turned out that the kind of play-doh we had exhibited quite a bit of beam-hardening, or significant X-ray energy dependence. After unsuccessful attempts to linearize the play-doh data we turned our attention to sand, which was assumed to have only negligible beam-hardening effects in the X-ray energies we use. Alas, there was still too much beam hardening with sand.
Next we found out that sugar produces a conveniently linear attenuation signal. We 3D-printed a compartmented holder, and filled the compartments one or two at the time with granulated sugar to produce temporal change in the target. Unfortunately, the holder showed up too strongly in the data.

In the end, our real $(2 + 1)$-dimensional spatio-temporal X-ray projection data was measured as follows. A set of sugar cubes were positioned into 10 different formations on a plate to create 10 different phantoms. Each of these stationary phantoms was measured using a cone-beam CT measurement system by taking a data set of 120 projection images with 3 degree angular step. From these data the fan-beam sinograms corresponding to the central slice of the sugar cubes were taken to serve as the test data.

In the spacetime level set reconstructions we use just 10 of those projection images (36 degree angular step), while the whole 120 image data set is only used for computing a ground truth reconstruction. The measurement geometry was the same for all the time steps, i.e. independent of time $t$.

The X-ray microtomography measurements of the sugar cube phantoms were performed with a custom-built μCT device nanotom supplied by PhoenixXray Systems + Services GmbH (Wunstorf, Germany). The X-ray detector used was a CMOS flat panel detector with $2304 \times 2284$ pixels of 50 μm size (Hamamatsu Photonics, Japan). Each projection image was composed of an average of eight 500 ms exposures. The X-ray tube acceleration voltage was 80 kV and tube current 225 μA.

5. Numerical results

Let us then look at the numerical results obtained by applying the three reconstruction methods of Section 3 to the tomographic data sets described in Section 4.

\[ \text{Original phantom} \quad \text{Proposed method, } n = 1 \quad \text{Proposed method, } n = 2 \quad \text{Tikhonov} \quad \text{Total variation} \]

\[ \begin{array}{cccc}
31\% & 31\% & 29\% & 30\% \\
26\% & 25\% & 25\% & 24\% \\
32\% & 32\% & 30\% & 34\% \\
28\% & 28\% & 27\% & 29\% \\
\end{array} \]

**Fig. 6.** Four different states (30th, 60th, 70th and 80th) and their reconstructions of the first simulated example. The images from left to right are: the original 2D phantom, reconstruction by the proposed method with $n = 1$, reconstruction by the proposed method with $n = 2$, non-negativity constrained Tikhonov reconstruction without temporal regularization and non-negativity constrained total variation reconstruction without temporal regularization. The relative errors of the reconstructions with respect to the 2-norm are given in the upper right corners of the reconstructions. Overall spatio-temporal resolution is $100 \times 100 \times 100$. The number of projection images taken at a single time step is seven ($7$). The regularization parameter $\alpha > 0$ was chosen separately for each method so that the relative errors of the reconstructions were (approximately) minimized.
Fig. 7. Three different states (30th, 60th and 95th) and their reconstructions of the second simulated example. The images from left to right are: the original 2D phantom, reconstruction by the proposed method with \( n = 1 \), reconstruction by the proposed method with \( n = 2 \), non-negativity constrained Tikhonov reconstruction without temporal regularization and non-negativity constrained total variation reconstruction without temporal regularization. The relative errors of the reconstructions with respect to the 2-norm are given in the upper right corners of the reconstructions. Overall spatio-temporal resolution is \( 100 \times 100 \times 100 \). The number of projection images taken at a single time step is seven [7]. The regularization parameter \( \alpha > 0 \) was chosen separately for each method so that the relative errors of the reconstructions were (approximately) minimized.

5.1. Simulated data

The Barzilai–Borwein optimization algorithm (with or without projection) was run for 50 iterations for each of the four reconstruction methods. The smoothing parameter for the proposed method with \( n = 2 \) was chosen to be \( \delta = 10^{-2} \) and the temporal spacing \( h_\ell = h_\ell \) equal to the spatial spacings, i.e. \( h_\ell = 1 \). The regularization parameter \( \alpha > 0 \) was chosen separately for each method such that the relative 2-norm errors of the reconstructions were (approximately) minimized. The reconstructions at four different stages for the first simulated example are shown in Fig. 6, and the reconstructions at three different stages for the second simulated example in Fig. 7.

5.2. Real X-ray data

Finally, we present the reconstructions computed from the measured X-ray data described in Section 4.2. The Barzilai–Borwein optimization algorithm (with or without projection) was run for 50 iterations for each of the four reconstruction methods. The smoothing parameter for the proposed method with \( n = 2 \) was chosen to be \( \delta = 10^{-4} \). The temporal spacing \( h_\ell = 50 \) was chosen to approximately equal to the approximate size of the changes in the 2D phantom from step to step. The regularization parameters \( \alpha > 0 \) were chosen separately for each method by visual inspection.

The first five of the ten 2D reconstructions of the sugar cubes are shown in Fig. 8 and the last five in Fig. 9. The ground truth reconstruction shown in the figures was computed from the larger set of 120 projection images using filtered back-projection algorithm.

6. Discussion and conclusions

The spacetime level set method is applicable to dynamic tomographic data collected using one or more source-detector pairs. One of the most promising frameworks for the method is a multisource arrangement where several fixed source-detector pairs are imaging a moving target at a high frame-rate. There are no moving parts in this setup, allowing easy calibration, simple construction and low maintenance.

The proposed method finds the reconstruction as a minimizer of a nonlinear functional motivated by level set methods. The level set type functional contains a regularization term penalizing the \( L^2 \) norms of up to \( n \in \{1, 2, \ldots\} \) derivatives of the reconstruction. We showed that in the case \( n = 1 \) the originally nonlinear and non-convex minimization problem reduces
to a rather standard convex Tikhonov problem that has a unique minimizer. For the case $n \geq 2$ we proved the existence of a minimizer under an assumption on the size of the regularization parameter and the signal-to-noise ratio.

Our computational experiments are based on both simulated and measured X-ray projection data of dynamic targets. The results suggest that the spacetime level set method with $n = 1$ and $n = 2$ are noise-robust, easy to implement and computationally effective. The reconstructions provided by the proposed method are significantly better than those provided by filtered back-projection and they also seem to outperform the non-negativity constrained Tikhonov regularization corresponding to the proposed method with $n = 1$ but with no temporal regularization. Compared to the non-negativity constrained total variation regularization without temporal regularization, there were only small differences in the results. In terms of the measured dynamic X-ray data, however, it would be interesting to test these methods for data sets with

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**Fig. 8.** First five states of the phantom and their reconstructions in the real data test case. The images from left to right are: ground truth, reconstruction by the proposed method with $n = 1$, reconstruction by the proposed method with $n = 2$ and non-negativity constrained total variation reconstruction without temporal regularization. The number of projection images at each instant is 10 and the spatial resolution is $256 \times 256$. 
Fig. 9. Last five states of the phantom and their reconstructions in the real data test case. The images from left to right are: ground truth, reconstruction by the proposed method with $n = 1$, reconstruction by the proposed method with $n = 2$ and non-negativity constrained total variation reconstruction without temporal regularization. The number of projection images at each instant is 10 and the spatial resolution is $256 \times 256$.

larger temporal resolution; these results would give more insight into the possible advantages of the temporal regularization incorporated in the proposed method.

In contrast to the classical binary-valued level set method, the proposed method models the X-ray attenuation coefficient by the level set function itself inside the level set. This allows us to prove powerful convergence results for the proposed method, which in turn show up as robustness and quick computational convergence. In particular, the classical level set method is known to require significant effort in tuning the parameters for a specific problem. Such tuning efforts are almost unnecessary in the proposed variant of the level set method.
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